Stability conditions and quivers

Tom Bridgeland

Spaces of stability conditions

Associated to any triangulated category D is a complex manifold $\mathrm{Stab}(D)$ whose points are stability conditions on D, i.e. pairs

$$Z\colon K_0(D) \to \mathbb{C}, \quad \mathcal{P} = \bigcup_{\phi \in \mathbb{R}} \mathcal{P}(\phi) \subset D,$$

satisfying some axioms.

(A) Do there exist stability conditions on $\mathrm{D^bCoh}(X)$ when $\dim_{\mathbb{C}}(X) \geqslant 3$?

(B) Given a fixed stability condition on D, can we construct good moduli stacks of semistable objects $E \in \mathcal{P}(\phi)$?

(C) Does the manifold $\mathrm{Stab}(D)$ carry any natural geometric structures (particularly in the CY_3 case)?

1. Quivers with potential

Quivers with potential

Let (Q, W) be a quiver with potential. Thus

(i) Q is an oriented graph,

(ii) W is a formal sum of oriented cycles in Q.

We always assume that Q has no loops or oriented 2-cycles.

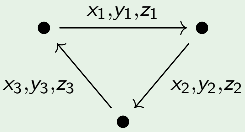
Associated to (Q, W) is a triangulated category $D^b(Q, W)$

By definition, $D^b(Q, W)$ is the subcategory of the derived category of the complete Ginzburg dg-algebra consisting of objects with finite-dimensional total cohomology.

Non-compact Calabi-Yau threefold

Example (Local $\mathbb{P}^2$)

Consider the quiver with potential



$$W = \sum_{i,j,k} \epsilon_{ijk} x_i y_j z_k .$$

Viewing the total space of the line bundle $\omega_{\mathbb{P}^2}$ as a non-compact Calabi-Yau threefold, there is an equivalence

$$D^b(Q, W) \cong D^b_{\mathbb{P}^2} \operatorname{Coh}(\omega_{\mathbb{P}^2}),$$

where on the right we consider the subcategory of objects supported on the zero-section.

General properties of $D = D^b(Q, W)$

(A) D has the CY_3 property:

$$\mathrm{Hom}^k(E, F) \cong \mathrm{Hom}^{3-k}(F, E)^*.$$

(B) D is generated by objects S_i indexed by the vertices of Q, and

$$\mathrm{Hom}^*(S_i, S_j) = \mathbb{C}^{\delta_{ij}} \oplus \mathbb{C}^{a_{ij}}[-1] \oplus \mathbb{C}^{a_{ji}}[-2] \oplus \mathbb{C}^{\delta_{ij}}[-3],$$

with a_{ij} the number of arrows in Q from vertex i to vertex j.

(C) There is a standard heart $\mathcal{A} \subset D$, which is finite-length, and whose simple objects are precisely the S_i.

Euler form and Poisson torus

Define $N = K_0(D) = \mathbb{Z}^{Q_0}$ and set

$$\mathbb{T} = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{C}^*) \cong (\mathbb{C}^*)^n.$$

The Euler form of D defines a skew-symmetric form

$$\langle -, - \rangle \colon N \times N \to \mathbb{Z},$$

$$\langle e_i, e_j \rangle = a_{ji} - a_{ij},$$

which induces an invariant Poisson structure on $\mathbb{T}$

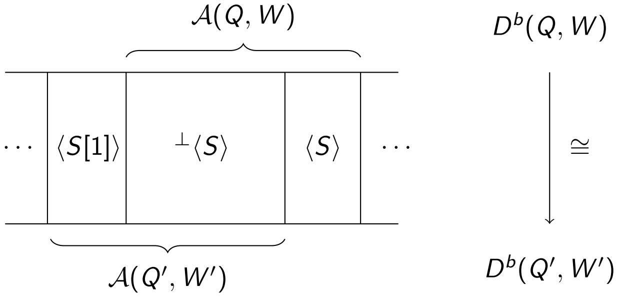
$$\{x^\alpha, x^\beta\} = \langle \alpha, \beta \rangle \cdot x^{\alpha+\beta}.$$

Tilting and mutation

Let (Q, W) be a QWP and choose a vertex i of Q. Write $S = S_i$.

$$\langle S \rangle = \{S^{\oplus n} : n \geqslant 0\} \subset \mathcal{A}, \quad {}^{\perp}\langle S \rangle = \{E \in \mathcal{A} : \mathrm{Hom}(E, S) = 0\}.$$

There is a mutation $(Q' W') = \mu_i(Q, W)$ and an equivalence



Exchange graphs

Let $D = D^b(Q, W)$ with W a generic potential.

(A) The heart exchange graph $\mathrm{EG}_\heartsuit(D)$ has

(I) vertices the finite-length hearts in D,

(II) edges connecting hearts related by a simple tilt.

(B) Each simple object S_i is spherical and defines an auto-equivalence Tw_{S_i}. The subgroup

$$\mathrm{Sph}(D) = \langle \mathrm{Tw}_{S_1}, \cdots, \mathrm{Tw}_{S_n} \rangle \subset \mathrm{Aut}(D)$$

is invariant under mutation.

(C) The cluster exchange graph of Q is the quotient

$$\mathrm{EG}(Q) = \mathrm{EG}_\heartsuit(D) / \mathrm{Sph}(D)$$

Stability space versus cluster variety

(A) For each heart $\mathcal{A} \in \mathrm{EG}_{\heartsuit}(D)$ there is a cell $\mathbb{H}^n \subset \mathrm{Stab}(D)$.

$$\bigcup_{\mathcal{A} \in \mathrm{EG}_{\heartsuit}(D)} \mathbb{H}^n \subset \mathrm{Stab}(D).$$

Note that the different cells only meet in their closures.

(B) The cluster variety is a union of tori glued by birational maps

$$\mathcal{X}(Q) = \bigcup_{\mathcal{A} \in \mathrm{EG}_{\heartsuit}(D)} \mathbb{T}.$$

$$x^{\beta} \mapsto x^{\beta} \cdot (1 + x^{\alpha})^{\langle \alpha, \beta \rangle}.$$

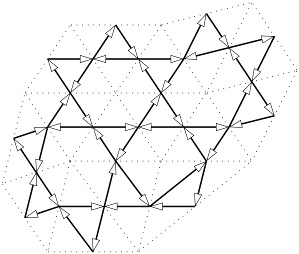
2. Examples from triangulated surfaces

From triangulations to quivers

Fix a surface S of genus g with a set $M = \{p_1, \cdots, p_d\} \subset S$.

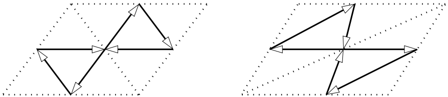
Consider triangulations of S with vertices at the points p_i.

Associated to any such triangulation is a quiver:



Flips and the exchange graph

A flip of the triangulation induces a mutation of the quiver:



(A) Fomin, Shapiro and Thurston proved that the cluster exchange graph is the set of (tagged) triangulations, with the edges corresponding to flips.

(B) Labardini-Fragoso dealt with the potentials associated to degenerate triangulations.

Space of stability conditions

Choose a generic potential W and set $D = D^b(Q, W)$.

Theorem (-, Ivan Smith)

$$\mathrm{Stab}(D)/\,\mathrm{Aut}(D) \cong \mathrm{Quad}(g, d).$$

The space $\mathrm{Quad}(g, d)$ parameterizes triples (S, M, ϕ) where

(A) S is a Riemann surface of genus g,

(B) $M = \sum_{i=1}^{d} p_i$ is a reduced divisor,

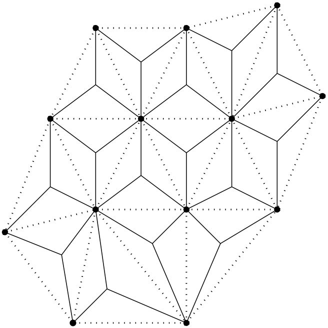
(C) $\phi \in H^0(S, \omega_S(M)^{\otimes 2})$ has simple zeroes.

Horizontal strip decomposition

A quadratic differential defines an unoriented foliation on S

$$\langle \sqrt{\phi(p)}, X \rangle \in \mathbb{R}, \qquad X \in T_p S.$$

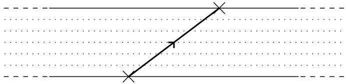
For a generic point $\phi \in \mathrm{Quad}(g, d)$ the trajectories split the surface into a disjoint union of cells known as horizontal strips.



Generic differentials define quivers

This leads to a triangulation and hence a quiver, together with a central charge function.

$$Z(S_i) = \int_{\gamma_i} \sqrt{\phi} \in \mathbb{C}.$$



When $Z(S_i)$ becomes real the triangulation degenerates and undergoes a flip. The heart of the corresponding stability condition undergoes a mutation.

Cluster variety

Let (S, M) be a marked surface as above, choose a triangulation and let Q be the corresponding quiver. Set $G = \mathrm{PGL}(2, \mathbb{C})$.

Theorem (Fock and Goncharov)

The cluster variety $\mathcal{X}(Q)$ is a dense open subset of the stack of labelled G-local systems on $S \setminus M$

$$\mathcal{X}(Q) \subset \mathrm{Loc}^*_G(S \setminus M) \xrightarrow{2^d:1} \mathrm{Loc}_G(S \setminus M).$$

The labelling is a choice of a monodromy-invariant section of the associated $\mathbb{P}^1$ bundle in a neighborhood of each marked point.

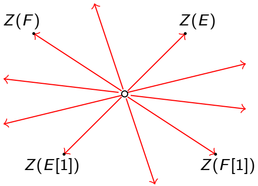
3. Donaldson-Thomas invariants

The active rays

For each stability condition $\sigma \in \mathrm{Stab}(D)$ there is a countable collection of active rays

$$\ell = \mathbb{R}_{>0} \exp(i\pi\phi) \subset \mathbb{C}$$

for which there exist semistable objects of phase ϕ.



As σ varies, the active rays move and may collide and separate.

Encoding DT invariants

To each active ray is associated a formal function on $\mathbb{T}$

$$\mathsf{DT}_\ell = \sum_{Z(\alpha)\in\ell} \mathsf{DT}_\sigma(\alpha)\cdot x^\alpha.$$

Ignoring convergence issues, there is a corresponding automorphism

$$S_\ell = \exp(\{\mathsf{DT}_\ell, -\}) \in \mathsf{Aut}(\mathbb{T})$$

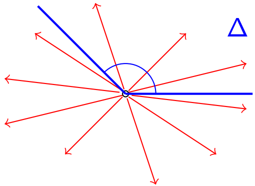
which is the time 1 Hamiltonian flow of the function DT_ℓ.

Wall-crossing formula

For any convex sector $\Delta \subset \mathbb{C}$, the clockwise product over active rays

$$\mathcal{S}_\Delta = \prod_{\ell \in \Delta} S_\ell \in \mathsf{Aut}(\mathbb{T})$$

remains constant as σ varies, providing no active ray crosses $\partial\Delta$.



This all makes good sense in a suitable completion $\mathbb{C}[[N_+]]$.

Example: the A_2 quiver

Let $\mathcal{A}$ be the abelian category of representations of the A_2 quiver. It has 3 indecomposable representations:

$$0 \longrightarrow S_2 \longrightarrow E \longrightarrow S_1 \longrightarrow 0.$$

We have $N = \mathbb{Z}^{\oplus 2} = \mathbb{Z}[S_1] \oplus \mathbb{Z}[S_2]$,

$$\langle (m_1, n_1), (m_2, n_2) \rangle = m_2 n_1 - m_1 n_2,$$

$$\mathbb{C}[N] = \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}] = \mathbb{C}[\mathbb{T}],$$

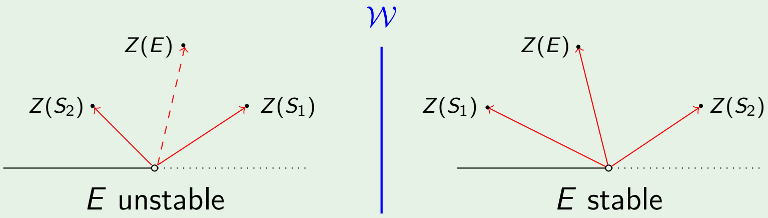
and the Poisson structure is

$$\{x_1, x_2\} = x_1 \cdot x_2.$$

Pentagon identity

The space $\mathrm{Stab}(\mathcal{A})$ is isomorphic to $\bar{\mathbb{H}}^2$, and there is a single wall

$$\mathcal{W} = \{Z \in \mathrm{Stab}(\mathcal{A}) : \mathrm{Im}\, Z(S_2)/Z(S_1) \in \mathbb{R}_{>0}\}$$



The wall-crossing formula is the cluster identity

$$C_{(0,1)} \circ C_{(1,0)} = C_{(1,0)} \circ C_{(1,1)} \circ C_{(0,1)}.$$

$$C_\alpha \colon x^\beta \mapsto x^\beta \cdot (1 + x^\alpha)^{\langle \alpha,\beta \rangle} \in \mathrm{Aut}\, \mathbb{C}[[x_1, x_2]].$$

4. Irregular connections and Stokes data

Stokes matrices and isomonodromy

The wall-crossing formula resembles an isomonodromy condition for an irregular connection with values in the infinite-dimensional group

$$G = \mathsf{Aut}_{\{-,-\}}(\mathbb{T})$$

of Poisson automorphisms of the torus $\mathbb{T} \cong (\mathbb{C}^*)^n$.

We first explain such phenomena in the finite-dimensional case, so set

$$G = \mathsf{GL}(n, \mathbb{C}), \quad \mathfrak{g} = \mathfrak{gl}(n, \mathbb{C}),$$

and introduce the decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}^{\mathrm{od}}, \quad \mathfrak{g}^{\mathrm{od}} = \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha, \quad \Phi = \{e_i^* - e_j^*\} \subset \mathfrak{h}^*.$$

A class of irregular connections

Consider meromorphic connections on the trivial G-bundle over the Riemann sphere $\mathbb{CP}^1$ of the form

$$\nabla = d - \left(\frac{U}{z^2} + \frac{V}{z}\right) dz,$$

(I) $U = \mathrm{diag}(u_1, \cdots, u_n) \in \mathfrak{h}$ is diagonal with distinct eigenvalues,

(II) $V \in \mathfrak{g}^{\mathrm{od}}$ has zeroes on the diagonal.

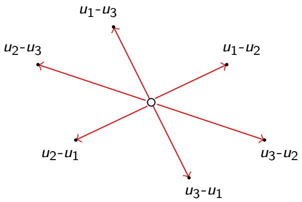
Then ∇ has an irregular singularity at 0 and a regular one at ∞.

The gauge equivalence class of a flat meromorphic connection with regular singularities is determined by its monodromy (Riemann-Hilbert correspondence). When irregular singularities are present one also needs to record Stokes data.

Stokes data of the connection

The Stokes rays for the connection ∇ are the rays

$$\mathbb{R}_{>0} \cdot (u_i - u_j) = \mathbb{R}_{>0} \cdot U(\alpha), \quad \alpha = e_i^* - e_j^*.$$



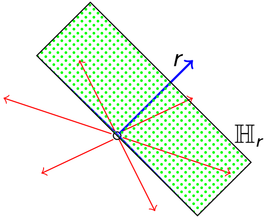
Associated to each Stokes ray ℓ is a Stokes factor

$$\mathcal{S}_\ell = \exp\Big(\sum_{U(\alpha)\in\ell} \epsilon_\alpha \Big) \in \exp\Big(\bigoplus_{U(\alpha)\in\ell} \mathfrak{g}_\alpha \Big) \subset G.$$

Canonical solution on a half-plane

Given a non-Stokes ray r, there is a canonical flat section X_r of ∇ on the orthogonal half-plane $\mathbb{H}_r$, uniquely defined by the condition that

$$X_r(t) \cdot e^{U/t} \to 1 \text{ as } t \to 0 \text{ in } \mathbb{H}_r.$$



As the ray r varies, the flat section X_r remains unchanged until r crosses a Stokes ray, where it jumps by

$$X_r \mapsto X_r \cdot S_\ell.$$

Iso-Stokes deformations

If we now vary the diagonal matrix U, we can deform the matrix V so that the Stokes factors remain constant. Such deformations are called iso-Stokes. More precisely:

For any convex sector $\Delta \subset \mathbb{C}^*$ the clockwise product

$$S_\Delta = \prod_{\ell \in \Sigma} S_\ell \in G,$$

remains constant unless a Stokes ray crosses the boundary of Σ.

Such variations are described by a system of partial differential equations giving the variation of V as a function of U.

5. Putting it together

Poisson vector fields on $\mathbb{T}$

Consider the group G of Poisson automorphisms of the torus

$$\mathbb{T} \cong \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{C}^*) \cong (\mathbb{C}^*)^n,$$

and the corresponding Lie algebra $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}^{\mathrm{od}}$, where

(A) the Cartan subalgebra

$$\mathfrak{h} = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{C}),$$

consists of translation-invariant vector fields on $\mathbb{T}$.

(B) the subspace $\mathfrak{g}^{\mathrm{od}}$ consists of Hamiltonian vector fields, and is the Poisson algebra of non-constant algebraic functions on $\mathbb{T}$

$$\mathfrak{g}^{\mathrm{od}} = \bigoplus_{\alpha \in N^{\times}} \mathfrak{g}_{\alpha} = \bigoplus_{\alpha \in N^{\times}} \mathbb{C} \cdot x^{\alpha}.$$

DT invariants as Stokes data

It is tempting to interpret the elements

$$S_\ell = \exp\left\{ \sum_{Z(\alpha)\in\ell} \mathrm{DT}_\sigma(\alpha)\cdot x^\alpha, - \right\} \in G$$

as defining Stokes factors for a G-valued connection of the form

$$\nabla = d - \left(\frac{Z}{t^2} + \frac{F}{t}\right) dt,$$

for some element $F \in \mathfrak{g}^{\mathrm{od}}$.

The wall-crossing formula is then precisely the condition that this family of connections is iso-Stokes as $\sigma \in \mathrm{Stab}(D)$ varies.

Iso-Stokes connection

Putting the canonical flat sections together should give a map

$$X\colon \operatorname{Stab}(D) \times \mathbb{C}^* \longrightarrow G.$$

Equivalently, setting $\mathcal{M} = \mathbb{T} \times \operatorname{Stab}(D)$, we expect a map

$$X\colon \mathcal{M} \times \mathbb{C}^* \longrightarrow \mathbb{T}.$$

The jumping behaviour means that the natural target is $\mathcal{X}(Q)$.

(I) How does this work in the cases coming from marked surfaces?
(II) Actually two versions (like Frobenius and tt^* in the $\mathrm{GL}(n)$ case).

Relating $\mathsf{Stab}(D)$ to $\mathcal{X}(Q)$

(1) Non-holomorphic version (Gaiotto-Moore-Neitzke):

$$\begin{array}{ccccc} & & \mathcal{M}^0_{Higgs} & \hookrightarrow & \mathcal{M}_{Betti} \cong \mathcal{X}(Q) \\ & & \downarrow (S^1)^n & & \\ \frac{\mathsf{Stab}(D)}{\mathsf{Aut}(D)} \cong \mathrm{Quad}(g,n) & \xleftarrow[\mathbb{C}-str.]{fix} & B_0 & & B_0 \subset H^0(S, K_S(D)^2) \end{array}$$

(2) Holomorphic version ('conformal limit'):

$$\frac{\mathsf{Stab}(D)}{\mathsf{Aut}(D)} \cong \mathrm{Quad}(g,n) \xrightarrow[non-canon.]{\cong} \mathrm{Proj}(g,n) \longrightarrow \mathcal{M}_{Betti} \cong \mathcal{X}(Q)$$

$$\mathrm{Quad}(g,n) \searrow \mathcal{M}(g,n) \swarrow \mathrm{Proj}(g,n)$$