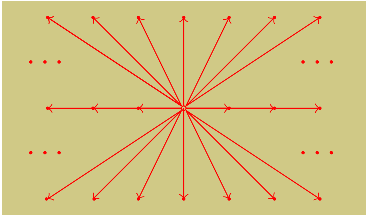
Wall-crossing for Donaldson-Thomas invariants

Tom Bridgeland

University of Sheffield



1. Stability conditions and wall-crossing

Generalized DT theory

Let (X, L) be a smooth, polarized projective CY_3 over $\mathbb{C}$.

Generalized (unrefined) DT theory (Joyce, Kontsevich-Soibelman) produces numbers $\mathsf{DT}_{X,L}(\gamma) \in \mathbb{Q}$ for classes $\gamma \in K_{num}(X)$.

They can be thought of as virtual Euler characteristics of the stack $\mathcal{M}_{X,L}(\gamma)$ of Gieseker semistable sheaves.

When there are no strictly semistables and $\mathcal{M}_{X,L}(\gamma)$ is smooth

$$\mathsf{DT}_{X,L}(\gamma) = (-1)^{\dim_{\mathbb{C}} M_{X,L}(\gamma)} \cdot e(M_{X,L}(\gamma)),$$

but in general the definition is much more complicated.

These numbers are invariant under deformations of (X, L), and satisfy an interesting wall-crossing formula as L is varied.

Stability conditions

A different context in which to study wall-crossing behaviour is provided by stability conditions on triangulated categories.

Let $\mathcal{D}$ be a triangulated category. A stability condition consists of

(I) A map of abelian groups $Z\colon K_0(\mathcal{D}) \to \mathbb{C}$,

(II) An $\mathbb{R}$-graded full subcategory $\mathcal{P} = \cup_{\phi\in\mathbb{R}}\mathcal{P}(\phi) \subset \mathcal{D}$,

together satisfying some axioms.

The map Z is called the central charge, and the objects of the subcategory $\mathcal{P}(\phi)$ are said to be semistable of phase ϕ.

Axioms for a stability condition

(A) if $0 \neq E \in \mathcal{P}(\phi)$ then $Z(E) \in \mathbb{R}_{>0} \cdot \exp(i\pi\phi)$,

(B) $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$ for all $\phi \in \mathbb{R}$,

(C) if $\phi_1 > \phi_2$ and $A_j \in \mathcal{P}(\phi_j)$ then $\mathrm{Hom}_{\mathcal{D}}(A_1, A_2) = 0$,

(D) for each $0 \neq E \in \mathcal{D}$ there is a finite collection of triangles

$$\begin{array}{ccccccccc} 0 = E_0 & \longrightarrow & E_1 & \longrightarrow \dots \longrightarrow & E_{n-1} & \longrightarrow & E_n = E \\ & \nwarrow \swarrow & & & & \nwarrow \swarrow & \\ & A_1 & & & & A_n & \end{array}$$

with $0 \neq A_j \in \mathcal{P}(\phi_j)$ and $\phi_1 > \phi_2 > \cdots > \phi_n$.

Stability manifold

Fix an abelian group homomorphism

$$\mathrm{ch}\colon K_0(\mathcal{D}) \to \Gamma \cong \mathbb{Z}^{\oplus n},$$

and insist that all central charges factor through ch.

Consider only stability conditions satisfying the support property:

$$\exists\, C > 0 \text{ such that } 0 \neq E \in \mathcal{P}(\phi) \implies |Z(E)| > C \cdot \|\,\mathrm{ch}(E)\|,$$

where $\|\cdot\|$ is a fixed norm on $\Gamma \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^n$.

Theorem

There is a complex manifold $\mathrm{Stab}(\mathcal{D})$ *whose points are the stability conditions on* $\mathcal{D}$*. The forgetful map defines a local homeomorphism*

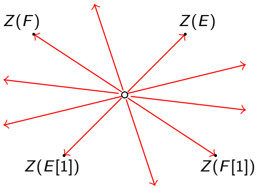
$$\mathrm{Stab}(\mathcal{D}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{C}) \cong \mathbb{C}^n.$$

Active rays

For each stability condition $\sigma \in \mathsf{Stab}(D)$ there is a countable collection of active rays

$$\ell = \mathbb{R}_{>0} \exp(i\pi\phi) \subset \mathbb{C}$$

for which there exist semistable objects of phase ϕ.



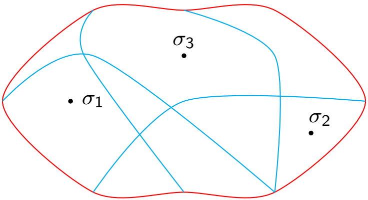
As σ varies, the active rays move and may collide and separate.

Wall-and-chamber structure

For a fixed class $\gamma \in \Gamma$, there is a locally-finite collection of real codimension one submanifolds

$$\mathcal{W} = \cup_\alpha \mathcal{W}_\alpha \subset \mathsf{Stab}(D)$$

such that the subcategory of semistable objects of class γ is constant in each connected component of the complement of $\mathcal{W}$.



DT invariants and wall-crossing

Assume that our triangulated category $\mathcal{D}$ satisfies the CY_3 property:

$$\mathrm{Hom}^i_{\mathcal{D}}(A, B) \cong \mathrm{Hom}^{3-i}_{\mathcal{D}}(B, A)^*.$$

In many examples there then exist generalized DT invariants

$$\mathrm{DT}_\sigma(\gamma) \in \mathbb{Q}, \qquad \gamma \in \Gamma \text{ and } \sigma \in \mathrm{Stab}(\mathcal{D})$$

associated to moduli spaces of σ-semistable objects of class γ.

Amazing fact (Joyce)

Knowing the full collection of invariants $\mathrm{DT}_\sigma(\gamma)$ at one point $\sigma \in \mathrm{Stab}(\mathcal{D})$ completely determines them at all other points.

Quivers with potential

When $\mathcal{D} = \mathcal{D}^b \operatorname{Coh}(X)$ with X a smooth projective Calabi-Yau threefold it is expected that Gieseker stability arises as a large volume limit of points in $\operatorname{Stab}(\mathcal{D})$.

But constructing stability conditions on $\mathcal{D}$ is very difficult.

A more tractable class of examples is provided by quivers with potential (Q, W). Recall

(i) Q is an oriented graph,

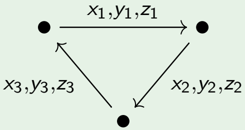
(ii) W is a $\mathbb{C}$-linear combination of oriented cycles in Q.

We always assume that Q has no loops or oriented 2-cycles.

Associated to (Q, W) is a triangulated category $D^b(Q, W)$

Local $\mathbb{P}^2$: a non-compact CY_3

Consider the quiver with potential



$$W = \sum_{i,j,k} \epsilon_{ijk} x_i y_j z_k.$$

Viewing the total space of the line bundle $\omega_{\mathbb{P}^2}$ as a non-compact Calabi-Yau threefold, there is an equivalence

$$D^b(Q, W) \cong D^b_{\mathbb{P}^2} \operatorname{Coh}(\omega_{\mathbb{P}^2}),$$

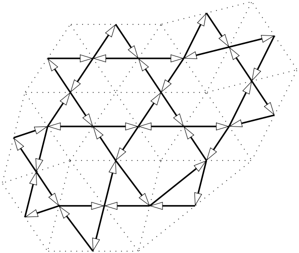
where on the right we consider the subcategory of objects supported on the zero-section.

Quivers from triangulations

Fix a surface S of genus g with a set $M = \{p_1, \cdots, p_d\} \subset S$.

Consider triangulations of S with vertices at the points p_i.

Associated to any such triangulation is a quiver:



Choose a generic potential W and set $\mathcal{D} = \mathcal{D}^b(Q, W)$.

Quadratic differentials

Theorem (-, Ivan Smith)

$$\mathrm{Stab}(\mathcal{D})/\mathrm{Aut}(\mathcal{D}) \cong \mathrm{Quad}(g, d).$$

The space $\mathrm{Quad}(g, d)$ parameterizes pairs (S, ϕ) with

(A) S is a Riemann surface of genus g,

(B) $D = \sum_{i=1}^{d} p_i$ is a reduced divisor,

(C) $\phi \in H^0(S, \omega_S(D)^{\otimes 2})$ has simple zeroes.

One can calculate DT invariants in these examples in terms of counts of finite-length trajectories of the corresponding quadratic differential.

2. BPS structures and the wall-crossing formula.

The output of (unrefined) DT theory

A BPS structure (Γ, Z, Ω) consists of

(A) An abelian group $\Gamma \cong \mathbb{Z}^{\oplus n}$ with a skew-symmetric form

$$\langle -, - \rangle \colon \Gamma \times \Gamma \to \mathbb{Z}$$

(B) A homomorphism of abelian groups $Z \colon \Gamma \to \mathbb{C}$,

(C) A map of sets $\Omega \colon \Gamma \to \mathbb{Q}$.

satisfying the conditions:

(I) Symmetry: $\Omega(-\gamma) = \Omega(\gamma)$ for all $\gamma \in \Gamma$,

(II) Support property: fixing a norm $\| \cdot \|$ on the finite-dimensional vector space $\Gamma \otimes_{\mathbb{Z}} \mathbb{R}$, there is a $C > 0$ such that

$$\Omega(\gamma) \neq 0 \implies |Z(\gamma)| > C \cdot \|\gamma\|.$$

Poisson algebraic torus

Consider the algebraic torus with character lattice Γ:

$$\mathbb{T}_+ = \mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{C}^*) \cong (\mathbb{C}^*)^n$$

$$\mathbb{C}[\mathbb{T}_+] = \bigoplus_{\gamma \in \Gamma} \mathbb{C} \cdot x_\gamma \cong \mathbb{C}[x_1^{\pm 1}, \cdots, x_n^{\pm n}].$$

The form $\langle -, - \rangle$ induces an invariant Poisson structure on $\mathbb{T}_+$:

$$\{x_\alpha, x_\beta\} = \langle \alpha, \beta \rangle \cdot x_\alpha \cdot x_\beta.$$

More precisely we should work with an associated torsor

$$\mathbb{T}_- = \big\{ g : \Gamma \to \mathbb{C}^* : g(\gamma_1 + \gamma_2) = (-1)^{\langle \gamma_1, \gamma_2 \rangle} g(\gamma_1) \cdot g(\gamma_2) \big\},$$

which we call the twisted torus.

DT Hamiltonians

The DT invariants $\mathrm{DT}(\gamma) \in \mathbb{Q}$ of a BPS structure are defined by

$$\mathrm{DT}(\gamma) = \sum_{\gamma = n\alpha} \frac{\Omega(\alpha)}{n^2}.$$

For any ray $\ell = \mathbb{R}_{>0} \cdot z \subset \mathbb{C}^*$ we consider the generating function

$$\mathrm{DT}(\ell) = \sum_{Z(\gamma) \in \ell} \mathrm{DT}(\gamma) \cdot x_\gamma.$$

A ray $\ell \subset \mathbb{C}^*$ is called active if this expression is nonzero.

We would like to think of the time 1 Hamiltonian flow of the function $\mathrm{DT}(\ell)$ as defining a Poisson automorphism $\mathrm{S}(\ell)$ of the torus $\mathbb{T}$.

Making sense of $\mathrm{S}(\ell)$

Formal approach

Restrict to classes γ lying in a positive cone $\Gamma^+ \subset \Gamma$, consider

$$\mathbb{C}[x_1^{\pm 1}, \cdots, x_n^{\pm 1}] \supset \mathbb{C}[x_1, \cdots, x_n] \subset \mathbb{C}[[x_1, \cdots, x_n]],$$

and the automorphism $\mathrm{S}(\ell)^* = \exp\{\mathsf{DT}(\ell), -\}$ of this completion.

Analytic approach

Restrict attention to BPS structures which are convergent:

$$\exists R > 0 \text{ such that } \sum_{\gamma \in \Gamma} |\Omega(\gamma)| \cdot e^{-R|Z(\gamma)|} < \infty.$$

Then on suitable analytic open subsets of $\mathbb{T}$ the sum $\mathsf{DT}(\ell)$ is absolutely convergent and its time 1 Hamiltonian flow $\mathrm{S}(\ell)$ exists.

Birational transformations

Often the maps $\mathrm{S}(\ell)$ are birational automorphisms of $\mathbb{T}$. Note

$$\exp\left\{\sum_{n\geq 1}\frac{x_{n\gamma}}{n^2},-\right\}(x_\beta)=x_\beta\cdot(1-x_\gamma)^{\langle\beta,\gamma\rangle}.$$

Whenever a ray $\ell\subset\mathbb{C}^*$ satisfies

(I) only finitely many active classes have $Z(\gamma_i)\in\ell$,

(II) these classes are mutually orthogonal $\langle\gamma_i,\gamma_j\rangle=0$,

(III) the corresponding BPS invariants $\Omega(\gamma_i)\in\mathbb{Z}$.

there is a formula

$$\mathrm{S}(\ell)^*(x_\beta)=\prod_{Z(\gamma)\in\ell}(1-x_\gamma)^{\Omega(\gamma)\cdot\langle\beta,\gamma\rangle}.$$

Variation of BPS structures

A framed variation of BPS structures over a complex manifold S is a collection of BPS structures (Γ, Z_s, Ω_s) indexed by $s \in S$ such that

(i) The numbers $Z_s(\gamma) \in \mathbb{C}$ vary holomorphically.

(ii) For any convex sector $\Delta \subset \mathbb{C}^*$ the clockwise ordered product

$$\mathrm{S}_s(\Delta) = \prod_{\ell \in \Delta} \mathrm{S}_s(\ell) \in \mathsf{Aut}(\mathbb{T})$$

is constant whenever the boundary of Δ remains non-active.

Part (ii) is the Kontsevich-Soibelman wall-crossing formula.

The complete set of numbers $\Omega_s(\gamma)$ at some point $s \in S$ determines them for all other points $s \in S$.

Example: the A_2 case

Let $\Gamma = \mathbb{Z}^{\oplus 2} = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ with $\langle e_1, e_2 \rangle = 1$. Then

$$\mathbb{C}[\mathbb{T}] = \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}], \quad \{x_1, x_2\} = x_1 \cdot x_2.$$

A central charge $Z \colon \Gamma \to \mathbb{C}$ is determined by $z_i = Z(e_i)$. Take

$$S = \mathfrak{h}^2 = \{(z_1, z_2) : z_i \in \mathfrak{h}\}.$$

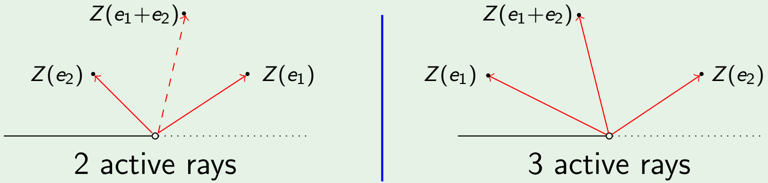
Define BPS invariants as follows:

(A) $\operatorname{Im}(z_2/z_1) > 0$. Set $\Omega(\pm e_1) = \Omega(\pm e_2) = 1$, all others zero.

(B) $\operatorname{Im}(z_2/z_1) < 0$. Set $\Omega(\pm e_1) = \Omega(\pm(e_1 + e_2)) = \Omega(\pm e_2) = 1$.

Wall-crossing formula: A_2 case

Two types of BPS structures appear, as illustrated below



The wall-crossing formula is the cluster pentagon identity

$$C_{(0,1)} \circ C_{(1,0)} = C_{(1,0)} \circ C_{(1,1)} \circ C_{(0,1)}.$$

$$C_\alpha \colon x_\beta \mapsto x_\beta \cdot (1 - x_\alpha)^{\langle \alpha, \beta \rangle}.$$

3. An analogy: iso-Stokes deformations of differential equations.

Stokes matrices and isomonodromy

The wall-crossing formula resembles an isomonodromy condition for an irregular connection with values in the infinite-dimensional group

$$G = \mathsf{Aut}_{\{-,-\}}(\mathbb{T})$$

of Poisson automorphisms of the torus $\mathbb{T} \cong (\mathbb{C}^*)^n$.

We first explain such phenomena in the finite-dimensional case, so set

$$G = \mathrm{GL}(n, \mathbb{C}), \quad \mathfrak{g} = \mathfrak{gl}(n, \mathbb{C}).$$

As a warm-up we start with the case of regular singularities.

A Fuchsian connection

We will consider meromorphic connections on the trivial G-bundle over the Riemann sphere $\mathbb{CP}^1$.

Consider a connection of the form

$$\nabla = d - \sum_{i=1}^{k} \frac{A_i \, dz}{z - a_i}$$

(I) $a_i \in \mathbb{C}$ are a set of k distinct points,

(II) $A_i \in \mathfrak{g}$ are corresponding residue matrices.

Then ∇ has regular singularities at the points a_i, and also at ∞.

Isomonodromic deformations

For each based loop

$$\gamma\colon S^1 \to \mathbb{C} \setminus \{a_1, \cdots, a_k\}$$

there is a corresponding monodromy matrix $\mathrm{Mon}_\gamma(\nabla) \in G$.

If we move the pole positions $a_i \in \mathbb{C}$, we can deform the residue matrices A_i so that all monodromy matrices remain constant. Such deformations are called isomonodromic.

Isomonodromic deformations are described by a system of partial differential equations: the Schlessinger equations.

A class of irregular connections

Introduce the decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}^{\mathrm{od}}, \quad \mathfrak{g}^{\mathrm{od}} = \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha, \quad \Phi = \{e_i^* - e_j^*\} \subset \mathfrak{h}^*.$$

Consider a connection of the form

$$\nabla = d - \left(\frac{U}{z^2} + \frac{V}{z} \right) dz,$$

(I) $U = \mathrm{diag}(u_1, \cdots, u_n) \in \mathfrak{h}$ is diagonal with distinct eigenvalues,

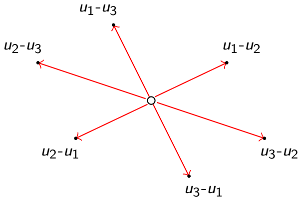
(II) $V \in \mathfrak{g}^{\mathrm{od}}$ has zeroes on the diagonal.

Then ∇ has an irregular singularity at 0 and a regular one at ∞.

Stokes data of the connection

The Stokes rays for the connection ∇ are the rays

$$\mathbb{R}_{>0} \cdot (u_i - u_j) = \mathbb{R}_{>0} \cdot U(\alpha), \quad \alpha = e_i^* - e_j^*.$$



We will associate to each Stokes ray ℓ a Stokes factor

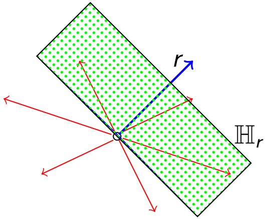
$$\mathrm{S}(\ell) = \exp\Big(\sum_{U(\alpha)\in\ell} \epsilon_\alpha \Big) \in \exp\Big(\bigoplus_{U(\alpha)\in\ell} \mathfrak{g}_\alpha \Big) \subset G.$$

Canonical solution on a half-plane

Theorem (Balser, Jurkat, Lutz)

Given a non-Stokes ray r, there is a unique flat section X_r of ∇ on the half-plane $\mathbb{H}_r \subset \mathbb{C}$ it spans, with the limiting property

$$X_r(t) \cdot e^{U/t} \to 1 \text{ as } t \to 0 \text{ in } \mathbb{H}_r.$$



Definition of Stokes factors

Suppose given two non-Stokes rays r_1, r_2 forming the boundary of a convex sector $\Delta \subset \mathbb{C}$. There is a unique $\mathrm{S}(\Delta) \in G$ with

$$X_{r_1}(t) = X_{r_2}(t) \cdot \mathrm{S}(\Delta), \qquad t \in \mathbb{H}_{r_1} \cap \mathbb{H}_{r_2}.$$

The defining property of $X_{r_i}(t)$ easily gives

$$\mathrm{S}(\Delta) \in \exp\Big(\bigoplus_{U(\alpha)\in\Delta} \mathfrak{g}_\alpha \Big) \subset G.$$

In particular $\mathrm{S}(\Delta) = 1$ if Δ contains no Stokes rays.

As the ray r varies, the canonical section X_r remains unchanged until r crosses a Stokes ray. The section then jumps by the Stokes factor

$$\mathrm{S}(\ell) = \exp\Big(\sum_{U(\alpha)\in\ell} \epsilon_\alpha \Big) \in \exp\Big(\bigoplus_{U(\alpha)\in\ell} \mathfrak{g}_\alpha \Big) \subset G.$$

Isomonodromy in the irregular case

If we now vary the diagonal matrix U, we can deform the matrix V so that the Stokes factors remain constant. Such deformations are called isomonodromic. More precisely:

For any convex sector $\Delta \subset \mathbb{C}^*$ the clockwise product

$$\mathrm{S}(\Delta) = \prod_{\ell \in \Delta} \mathrm{S}(\ell) \in G,$$

remains constant unless a Stokes ray crosses the boundary of Δ.

Isomonodromic variations are again described by a system of partial differential equations.

Poisson vector fields on $\mathbb{T}$

Consider the group G of Poisson automorphisms of the torus

$$\mathbb{T} \cong \mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{C}^*) \cong (\mathbb{C}^*)^n,$$

and the corresponding Lie algebra $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}^{\mathrm{od}}$, where

(A) the Cartan subalgebra

$$\mathfrak{h} = \mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{C}),$$

consists of translation-invariant vector fields on $\mathbb{T}$.

(B) the subspace $\mathfrak{g}^{\mathrm{od}}$ consists of Hamiltonian vector fields, and is the Poisson algebra of non-constant algebraic functions on $\mathbb{T}$

$$\mathfrak{g}^{\mathrm{od}} = \bigoplus_{\gamma \in \Gamma \setminus \{0\}} \mathfrak{g}_\gamma = \bigoplus_{\gamma \in \Gamma \setminus \{0\}} \mathbb{C} \cdot x_\gamma.$$

DT invariants as Stokes data

It is tempting to interpret the elements

$$\mathrm{S}(\ell) = \exp\Bigg(\sum_{Z(\gamma)\in\ell} \mathsf{DT}_\sigma(\gamma)\cdot x_\gamma\Bigg) \in G$$

as defining Stokes factors for a G-valued connection of the form

$$\nabla = d - \left(\frac{Z}{t^2} + \frac{F}{t}\right) dt,$$

where $F \in \mathfrak{g}^{\mathrm{od}}$ depends holomorphically on Z, or equivalently σ.

The wall-crossing formula is precisely the condition that this family of connections is isomonodromic as $\sigma \in \mathrm{Stab}(\mathcal{D})$ varies.

How to calculate F?

We know the Stokes factors $\mathrm{S}(\ell)$ and would like to find $F = F(Z)$.

To do this we should first find the canonical solutions $X_r(t)$.

We can assemble these to make a piecewise holomorphic function

$$X \colon \mathbb{C}^* \to G = \mathrm{Aut}_{\{-,-\}}(\mathbb{T}).$$

This satisfies a Riemann-Hilbert problem: it has known behaviour as $t \to 0$ and $t \to \infty$, and prescribed jumps as t crosses a Stokes ray.

Rather than working with the infinite-dimensional group G, we fix a point $\xi \in \mathbb{T}$ and compose X_r with the map $\mathrm{eval}_\xi \colon G \to \mathbb{T}$ to get

$$\Phi \colon \mathbb{C}^* \to \mathbb{T}.$$

4. The Riemann-Hilbert problem.

The Riemann-Hilbert problem

Fix a BPS structure (Γ, Z, Ω) and a point $\xi \in \mathbb{T}$.

Find a piecewise holomorphic function $\Phi\colon \mathbb{C}^* \to \mathbb{T}$ satisfying:

(I) (Jumping): When t crosses an active ray ℓ clockwise,

$$\Phi(t) \mapsto \mathrm{S}(\ell)(\Phi(t)).$$

(II) (Limit at 0): Write $\Phi_\gamma(t)) = x_\gamma(\Phi(t))$. As $t \to 0$,

$$\Phi_\gamma(t) \cdot e^{Z(\gamma)/t} \to x_\gamma(\xi).$$

(III) (Growth at ∞): For any $\gamma \in \Gamma$ there exists $k > 0$ with

$$|t|^{-k} < |\Phi_\gamma(t)| < |t|^k \text{ as } t \to \infty.$$

THE A_1 EXAMPLE

Consider the following BPS structure

(I) The lattice $\Gamma = \mathbb{Z} \cdot \gamma$ is one-dimensional. Thus $\langle -, - \rangle = 0$.

(II) The central charge $Z \colon \Gamma \to \mathbb{C}$ is determined by $z = Z(\gamma) \in \mathbb{C}^*$,

(III) The only non-vanishing BPS invariants are $\Omega(\pm\gamma) = 1$.

Then $\mathbb{T} = \mathbb{C}^*$ and all automorphisms $\mathrm{S}(\ell)$ are the identity.

$$\Phi_\gamma(t) = \xi \cdot \exp(-z/t) \in \mathbb{T} = \mathbb{C}^*.$$

Now double the BPS structure: take the lattice $\Gamma \oplus \Gamma^\vee$ with canonical skew form, and extend Z and Ω by zero. Consider

$$y(t) = \Phi_{\gamma^\vee}(t) \colon \mathbb{C}^* \to \mathbb{C}^*.$$

Doubled A_1 case

Consider the case $\xi = 1$. The map $y\colon \mathbb{C}^* \to \mathbb{C}^*$ should satisfy

(I) y is holomorphic away from the rays $\mathbb{R}_{>0} \cdot (\pm z)$ and has jumps

$$y(t) \mapsto y(t) \cdot (1 - x(t)^{\pm 1})^{\pm 1}, \quad x(t) = \exp(-z/t),$$

as t moves clockwise across them.

(II) $y(t) \to 1$ as $t \to 0$.

(III) there exists $k > 0$ such that

$$|t|^{-k} < |y(t)| < |t|^k \text{ as } t \to \infty.$$

Solution: the Gamma function

The doubled A_1 problem has the unique solution

$$y(t) = \Delta\left(\frac{\pm z}{2\pi i t}\right)^{\mp 1} \quad \text{where} \quad \Delta(w) = \frac{e^w \cdot \Gamma(w)}{\sqrt{2\pi} \cdot w^{w-\frac{1}{2}}},$$

in the half-planes $\pm \operatorname{Im}(t/z) > 0$.

This is elementary: all you need is

$$\Gamma(w) \cdot \Gamma(1-w) = \frac{\pi}{\sin(\pi w)}, \quad \Gamma(w+1) = w \cdot \Gamma(w),$$

$$\log \Delta(w) \sim \sum_{g=1}^{\infty} \frac{B_{2g}}{2g(2g-1)} w^{1-2g}.$$

The tau function

Suppose given a framed variation of BPS structures (Γ, Z_p, Ω_p) over a complex manifold S such that

$$\pi\colon S \to \mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{C}) = \mathbb{C}^n, \quad s \mapsto Z_s,$$

is a local isomorphism. Taking a basis $(\gamma_1, \cdots, \gamma_n) \subset \Gamma$ we get local co-ordinates $z_i = Z_s(\gamma_i)$ on S.

Suppose we are given analytically varying solutions $\Phi_\gamma(z_i, t)$ to the Riemann-Hilbert problems associated to (Γ, Z_s, Ω_s).

Define a function $\tau = \tau(z_i, t)$ by the relation

$$\frac{\partial}{\partial t} \log \Phi_{\gamma_k}(z_i, t) = \sum_{j=1}^{n} \epsilon_{jk} \frac{\partial}{\partial z_j} \log \tau(z_i, t), \quad \epsilon_{jk} = \langle \gamma_j, \gamma_k \rangle.$$

Solution in uncoupled case

In the A_1 case the τ-function is essentially the Barnes G-function.

$$\log \tau(z,t) \sim \sum_{g\geq 1} \frac{B_{2g}}{2g(2g-2)} \left(\frac{2\pi i t}{z}\right)^{2g-2}.$$

Whenever our BPS structures are uncoupled

$$\Omega(\gamma_i) \neq 0 \implies \langle \gamma_1, \gamma_2 \rangle = 0,$$

we can try to solve the RH problem by superposition of A_1 solutions. This works precisely if only finitely many $\Omega(\gamma) \neq 0$.

$$\log \tau(z,t) \sim \sum_{g\geq 1} \sum_{\gamma\in\Gamma} \frac{\Omega(\gamma) \cdot B_{2g}}{2g(2g-2)} \left(\frac{2\pi i t}{Z(\gamma)}\right)^{2g-2}$$

Geometric case: curves on a CY_3

Can apply this to coherent sheaves on a compact Calabi-Yau threefold supported in dimension ≤ 1. We have

$$\Gamma = H_2(X, \mathbb{Z}) \oplus \mathbb{Z}, \quad Z(\beta, n) = 2\pi(\beta \cdot \omega_{\mathbb{C}} - n).$$

$$\Omega(\beta, n) = \mathrm{GV}_0(\beta), \quad \Omega(0, n) = -\chi(X).$$

Since $\chi(-,-) = 0$ these BPS structures are uncoupled.

$$\tau(\omega_{\mathbb{C}}, t) \overset{pos.\ deg}{\sim} \sum_{g \geq 2} \frac{\chi(X)\, B_{2g}\, B_{2g-2}}{4g\,(2g-2)\,(2g-2)!} \cdot (2\pi t)^{2g-2}$$

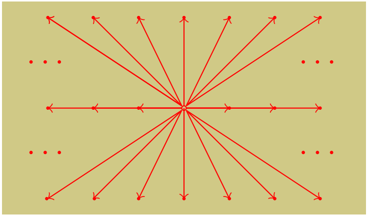
$$+ \sum_{\beta \in H_2(X,\mathbb{Z})} \sum_{k \geq 1} \mathrm{GV}_0(\beta) \frac{e^{2\pi i \omega \cdot k\beta}}{4k} \sin^{-2}(i\pi t k).$$

This matches the contribution to the topological string partition function of the genus 0 GV invariants.

Example: conifold BPS structure

Applying DT theory to the resolved conifold gives a variation of BPS structures over the space

$$\left\{(v, w) \in \mathbb{C}^2 : w \neq 0 \text{ and } v + dw \neq 0 \text{ for all } d \in \mathbb{Z}\right\} \subset \mathbb{C}^2.$$



It is given by $\Gamma = \mathbb{Z}^{\oplus 2}$ with $\langle -, - \rangle = 0$, $Z(r, d) = rv + dw$ and

$$\Omega(\gamma) = \begin{cases} 1 & \text{if } \gamma = \pm(1, d) \text{ for some } d \in \mathbb{Z}, \\ -2 & \text{if } \gamma = (0, d) \text{ for some } 0 \neq d \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}$$

Non-perturbative partition function

The corresponding RH problems have unique solutions, which can be written explicitly in terms of Barnes double and triple sine functions.

$$\tau(v,w,t) = H(v,w,t)\cdot\exp(R(v,w,t)),$$

$$H(v,w,t) = \exp\left(\int_{\mathbb{R}+i\epsilon}\frac{e^{vs}-1}{e^{ws}-1}\cdot\frac{e^{ts}}{(e^{ts}-1)^2}\cdot\frac{ds}{s}\right),$$

$$R(v,w,t) = \left(\frac{w}{2\pi it}\right)^2\left(\mathrm{Li}_3(e^{2\pi iv/w})-\zeta(3)\right)+\frac{i\pi}{12}\cdot\frac{v}{w}.$$

The function H is a non-perturbative closed-string partition function.