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Algebra/Homological Algebra

A criterion for regularity of local rings

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Abstract

It is proved that a noetherian commutative local ring A containing a field is regular if there is a complex M of free A-modules with the following properties: $M_i = 0$ for $i \notin [0, \dim A]$; the homology of M has finite length; $\mathrm{H}_0(M)$ contains the residue field of A as a direct summand. This result is an essential component in the proofs of the McKay correspondence in dimension 3 and of the statement that threefold flops induce equivalences of derived categories. ***To cite this article: T. Bridgeland, S. Iyengar, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Résumé

Une critère pour la régularité des anneaux locaux. On démontre qu'un anneau local noethérien commutatif A contenant un corps est régulier s'il existe un complexe M de A-modules libres avec les propriétés suivantes : $M_i = 0$ pour $i \notin [0, \dim A]$; l'homologie de M est de longueur finie ; $\mathrm{H}_0(M)$ contient le corps résiduel de A en tant que facteur direct. Ce résultat est une composante essentielle dans les démonstrations de la correspondance de McKay en dimension 3 et du fait que les flops de dimension trois induisent des équivalences de catégories dérivées. ***Pour citer cet article : T. Bridgeland, S. Iyengar, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

1. Introduction

Let $(A, \mathfrak{m}, k)$ be a local ring; thus A is a noetherian commutative local ring, with maximal ideal $\mathfrak{m}$ and residue field k. Let $M : 0 \to M_d \to \cdots \to M_0 \to 0$ be a complex of free A-modules with $\operatorname{length}_A \mathrm{H}(M)$ finite and non-zero. The New Intersection Theorem [10] yields $d \geqslant \dim A$.

In this Note we prove the following result, which is akin to Serre's theorem that a local ring is regular when its residue field has a finite free resolution:

Theorem 1.1. *Assume A contains a field or $\dim A \leqslant 3$. If $d = \dim A$ and k is a direct summand of $\mathrm{H}_0(M)$, then $\mathrm{H}_i(M) = 0$ for $i \geqslant 1$, and the local ring A is regular.*

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This result is contained in Theorem 2.4. A restricted version of such a statement occurs as [3, (4.3)]; however, as is explained in the last paragraph of this article, the proof of [3, (4.3)] is incorrect.

In the remainder of the introduction, the field k is algebraically closed, schemes over k are of finite type, and the points considered are closed points. We write $\mathrm{D}(Y)$ for the bounded derived category of coherent sheaves on a scheme Y, and k_y for the structure sheaf of a point $y \in Y$. Given the theorem above, arguing as in [3, §5], one obtains

Corollary 1.2. *Let Y be an irreducible scheme of dimension d over k, and let E be an object of $\mathrm{D}(Y)$. Suppose there is a point $y_0 \in Y$ such that k_{y_0} is a direct summand of $\mathrm{H}_0(E)$ and*

$$\mathrm{Hom}^i_{\mathrm{D}(Y)}(E, k_y) = 0 \quad \text{unless } y = y_0 \text{ and } 0 \leqslant i \leqslant d.$$

Then Y is non-singular at y_0 and $E \cong \mathrm{H}_0(E)$ in $\mathrm{D}(Y)$.

This result enables one to show that certain moduli spaces are non-singular and give rise to derived equivalences; see [3, (6.1)]. This has proved particularly effective in dimension three, and is an essential component in the proofs in [1,2].

2. Proof of the main theorem

This section is dedicated to a proof Theorem 2.4. The book of Bruns and Herzog [4] is our standard reference for the notions that appear here.

Let $(A, \mathfrak{m}, k)$ be a local ring, and let C be an A-module; it need not be finitely generated. A sequence $\boldsymbol{x} = x_1, \ldots, x_n$ is *C-regular* if $\boldsymbol{x}C \neq C$ and x_i is a non-zero-divisor on $C/(x_1, \ldots, x_{i-1})C$ for each $1 \leqslant i \leqslant n$. An A-module C is *big Cohen–Macaulay* if there is a system of parameters for A that is C-regular. If every system of parameters of A is C-regular, then C is said to be *balanced*. Any ring that has a big Cohen–Macaulay module has also one that is balanced; see [4, (8.5.3)]. Big Cohen–Macaulay modules were introduced by Hochster [6], who constructed them when A contains a field, and, following the recent work of Heitmann, also when $\dim A \leqslant 3$; see [7].

The result below is contained in the proof of [5, (1.13)] by Evans and Griffith, see also [8, (3.1)], so only an outline of an argument is provided; it follows the discussion around [4, (9.1.7)]; see also [8, (3.4)].

Lemma 2.1. *Let M be a complex of free A-modules with $M_i = 0$ for $i \notin [0, \dim A]$, the A-module $\mathrm{H}_0(M)$ finitely generated, and $\mathrm{length}_A\, \mathrm{H}_i(M)$ finite for each $i \geqslant 1$.*

If C is a balanced big Cohen–Macaulay A-module, then $\mathrm{H}_i(M \otimes_A C) = 0$ for $i \geqslant 1$.

Sketch of a proof. Replacing M by a quasi-isomorphic complex we may assume each M_i is finite free. Choose a basis for each M_i and let ϕ_i be the matrix representing the differential $\partial_i : M_i \to M_{i-1}$. Set $r_i = \sum_{j=i}^{n} (-1)^{j-i} \mathrm{rank}_A M_j$, and let $I_{r_i}(\phi_i)$ be the ideal generated by the $r_i \times r_i$ minors of ϕ_i. Fix an integer $1 \leqslant i \leqslant d$, and let $\mathfrak{p}$ be a prime ideal of A with $\mathrm{height}\, I_{r_i}(\phi_i) = \mathrm{height}\, \mathfrak{p}$. If $\mathrm{height}\, \mathfrak{p} = \dim A$, then $\mathrm{height}\, I_{r_i}(\phi_i) = \dim A \geqslant d \geqslant i$, where the first inequality holds by hypotheses. If $\mathrm{height}\, \mathfrak{p} < \dim A$, then $\mathrm{H}_j(M \otimes_A A_\mathfrak{p}) = 0$ for $j \geqslant 1$, by hypotheses. Therefore

$$\mathrm{height}\, I_{r_i}(\phi_i) = \mathrm{height}\big(I_{r_i}(\phi_i) \otimes_A A_\mathfrak{p}\big) = \mathrm{height}\, I_{r_i}(\phi_i \otimes_A A_\mathfrak{p}) \geqslant i,$$

where the inequality is comes from the Buchsbaum–Eisenbud acyclicity criterion [4, (9.1.6)]. Thus, no matter what $\mathrm{height}\, \mathfrak{p}$ is, one has

$$\dim A - \dim A/I_{r_i}(\phi_i) \geqslant \mathrm{height}\, I_{r_i}(\phi_i) \geqslant i.$$

Thus, $I_{r_i}(\phi_i)$ contains a sequence $\boldsymbol{x} = x_1, \ldots, x_i$ that extends to a full system of parameters for R. Since C is balanced big Cohen–Macaulay module, the sequence $\boldsymbol{x}$ is C-regular, so another application of [4, (9.1.6)] yields the desired result. □

The following elementary remark is invoked twice in the arguments below.

Lemma 2.2. *Let R be a commutative ring and U, V complexes of R-modules with $U_i = 0 = V_i$ for each $i < 0$. If each R-module V_i is flat and $\mathrm{H}_1(U \otimes_R V) = 0$, then $\mathrm{H}_1(\mathrm{H}_0(U) \otimes_R V) = 0$.*

Proof. Let $\widetilde{U} = \text{Ker}(U \to \text{H}_0(U))$; evidently, $\text{H}_i(\widetilde{U}) = 0$ for $i < 1$. Since $- \otimes_R V$ preserves quasi-isomorphisms, $\text{H}_i(\widetilde{U} \otimes_R V) = 0$ for $i < 1$. The long exact sequence that results from the short exact sequence of complexes $0 \to \widetilde{U} \to U \to \text{H}_0(U) \to 0$ thus yields a surjective homomorphism

$$\text{H}_1(U \otimes_R V) \longrightarrow \text{H}_1\big(\text{H}_0(U) \otimes_R V\big) \longrightarrow 0.$$

This justifies the claim. □

The proposition below is immediate when C is finitely generated: $\text{Tor}_1^A(C,k) = 0$ implies C is free. It contains [12, (2.5)], due to Schoutens, which deals with the case when C is a big Cohen–Macaulay algebra.

Proposition 2.3. *Let $(A, \mathfrak{m}, k)$ be a local ring and C an A-module with $\mathfrak{m}C \neq C$. If $\text{Tor}_1^A(C,k) = 0$, then each C-regular sequence is A-regular.*

Proof. First we establish that for any ideal I in A one has $(IC :_R C) = I$.

Indeed, consider first the case where the ideal I is $\mathfrak{m}$-primary ideal.

Let F be a flat resolution of C as an A-module, and let V be a flat resolution of k, viewed as an A/I-module. Therefore $\text{H}_1(F \otimes_A V) = \text{Tor}_1^A(C,k) = 0$. The A action on k factors through A/I, so $F \otimes_A V \cong (F \otimes_A A/I) \otimes_{A/I} V$. Thus, applying Lemma 2.2 with $U = C \otimes_A A/I$ one obtains

$$\text{Tor}_1^{A/I}(C/IC, k) = \text{H}_1\big(\text{H}_0(U) \otimes_{A/I} V\big) = 0.$$

The ring A/I is artinian and local, with residue field k, thus $\text{Tor}_1^{A/I}(C/IC,k) = 0$ implies that the A/I-module C/IC is free; see, for instance, [9, (22.3)]. Moreover, C/IC is non-zero as $\mathfrak{m}C \neq C$. Thus $(IC :_R C) = I$, as desired.

For an arbitrary ideal I, evidently $I \subseteq (IC :_R C)$. The reverse inclusion follows from the chain:

$$(IC :_R C) \subseteq \bigcap_{n \in \mathbb{N}} \big((I + \mathfrak{m}^n)C :_R C\big) = \bigcap_{n \in \mathbb{N}} \big(I + \mathfrak{m}^n\big) = I,$$

where the first equality holds because each ideal $(I + \mathfrak{m}^n)$ is a $\mathfrak{m}$-primary, while the second one is by the Krull Intersection Theorem. This settles the claim.

Let $x_1, \ldots, x_m$ be a regular sequence on C. Fix an integer $1 \leqslant i \leqslant m$, and set $I = (x_1, \ldots, x_{i-1})$. For any element r in A, the first and the second implications below are obvious:

$$rx_i \in I \implies rx_iC \subseteq IC \implies x_i(rC) \subseteq IC \implies rC \subseteq IC \implies r \in I.$$

The third implication holds because x_i is regular on C/IC, and the last one is by the claim established above. Thus, x_i is a non-zero-divisor on A/I, that is to say, on $A/(x_1, \ldots, x_{i-1})$. Since this holds for each i, we deduce that the sequence $\boldsymbol{x}$ is regular on A, as desired. □

The result below contains the theorem stated in the introduction.

Theorem 2.4. *Let $(A, \mathfrak{m}, k)$ be a local ring, M a complex of free A-modules with $M_i = 0$ for $i \notin [0, \dim A]$, the A-module $\text{H}_0(M)$ finitely generated, and $\text{length}_A \text{H}_i(M)$ finite for $i \geqslant 1$. Assume A has a big Cohen–Macaulay module. If k is a direct summand of $\text{H}_0(M)$, then the local ring A is regular.*

Proof. Let C be a big Cohen–Macaulay A-module; we may assume that C is balanced. Let $d = \dim A$, and let $\boldsymbol{x} = x_1, \ldots, x_d$ be a system of parameters for A that is a regular sequence on C. In particular, $\boldsymbol{x}C \neq C$, and hence $\mathfrak{m}C \neq C$, since $\boldsymbol{x}$ is $\mathfrak{m}$-primary.

Let V be a flat resolution of C over A. The complex M is finite and consists of free modules, so $\text{H}(M \otimes_A C) \cong \text{H}(M \otimes_A V)$, and hence $\text{H}_1(M \otimes_A V) = 0$, by Lemma 2.1. Now Lemma 2.2, invoked with $U = M$ implies $\text{H}_1(\text{H}_0(M) \otimes_A V) = 0$, so $\text{Tor}_1^A(\text{H}_0(M), C) = 0$. Since k is a direct summand of $\text{H}_0(M)$, this implies $\text{Tor}_1^A(C,k) = \text{Tor}_1^A(k,C) = 0$.

By Proposition 2.3, the sequence $\boldsymbol{x}$ is A-regular, therefore $\operatorname{depth} A \geqslant \dim A$ and A is (big) Cohen–Macaulay. Consequently, Lemma 2.1, now applied with $C = A$, implies $\text{H}_i(M) = 0$ for $i \geqslant 1$, as claimed. In particular, M is a finite free resolution of $\mathrm{H}_0(M)$, so the projective dimension of $\mathrm{H}_0(M)$ is finite. Therefore, the projective dimension of k is finite as well, since it is a direct summand of $\mathrm{H}_0(M)$. Thus, A is regular; see Serre [11, Ch. IV, cor. 2, th. 9]. □

As stated in the introduction, the proof of [3, (4.3)] is incorrect; in the paragraph below we adopt the notation of [3, (4.3)]. The problem with it is the claim in display (1), on [3, pg. 639], which reads:

$$\mathrm{Tor}_p^A(N, C) = 0 \quad \text{for all } p > 0. \tag{$*$}$$

This cannot hold unless we assume a priori that the local ring A is Cohen–Macaulay.

Indeed, suppose the displayed claim is true. Consider the standard change of rings spectral sequence sitting in the first quadrant:

$$\mathrm{E}^2_{p,q} = \mathrm{Tor}_p^A\big(\mathrm{Tor}_q^R(k, A), C\big) \Longrightarrow \mathrm{Tor}^R_{p+q}(k, C).$$

The edge homomorphisms in the spectral sequence give rise to the exact sequence

$$0 = \mathrm{Tor}_2^A(N, C) \longrightarrow \mathrm{Tor}_1^R(k, A) \otimes_A C \longrightarrow \mathrm{Tor}_1^R(k, C) = 0,$$

where the 0 on the left holds by $(*)$ and that on the right holds because C is free over R. Thus, the middle term is 0, which implies $\mathrm{Tor}_1^R(k, A) = 0$. Therefore A is free as an R-module, because A is finitely generated over R, and hence A is Cohen–Macaulay.

References

[1] T. Bridgeland, Flops and derived categories, Invent. Math. 147 (2002) 613–632.
[2] T. Bridgeland, A. King, M. Reid, The McKay correspondence as an equivalence of derived categories, J. Amer. Math. Soc. 14 (2001) 535–554.
[3] T. Bridgeland, A. Maciocia, Fourier–Mukai transforms for K3 and elliptic fibrations, J. Alg. Geom. 11 (2002) 629–657.
[4] W. Bruns, J. Herzog, Cohen–Macaulay Rings, Cambridge Stud. Adv. Math., vol. 39, Cambridge Univ. Press, 1998.
[5] E.G. Evans, P. Griffith, Syzygies, London Math. Soc. Lecture Note Ser., vol. 106, Cambridge Univ. Press, 1985.
[6] M. Hochster, Topics in the Homological Theory of Modules Over Commutative Rings, CBMS Reg. Conf. Ser. Math., vol. 24, Amer. Math. Soc., 1975.
[7] M. Hochster, Big Cohen–Macaulay algebras in dimension three via Heitmann's theorem, J. Algebra 254 (2002) 395–408.
[8] S. Iyengar, Depth for complexes, and intersection theorems, Math. Z. 230 (1999) 545–567.
[9] H. Matsumura, Commutative Ring Theory, Cambridge Stud. Adv. Math., vol. 8, Cambridge Univ. Press, Cambridge, 1986.
[10] P. Roberts, Le théorème d'intersection, C. R. Acad. Sci. Paris, Ser. I 304 (1987) 177–180.
[11] J.-P. Serre, Local Algebra, Springer, 2000.
[12] H. Schoutens, On the vanishing of Tor of the absolute integral closure, J. Algebra 275 (2004) 567–574.