FOURIER-MUKAI TRANSFORMS FOR K3 AND ELLIPTIC FIBRATIONS

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Abstract. Given a non-singular variety with a K3 fibration $\pi\colon X \to S$ we construct dual fibrations $\hat{\pi}\colon Y \to S$ by replacing each fibre X_s of π by a two-dimensional moduli space of stable sheaves on X_s. In certain cases we prove that the resulting scheme Y is a non-singular variety and construct an equivalence of derived categories of coherent sheaves $\Phi\colon \mathrm{D}(Y) \to \mathrm{D}(X)$. Our methods also apply to elliptic and abelian surface fibrations. As an application we use the equivalences Φ to relate moduli spaces of stable bundles on elliptic threefolds to Hilbert schemes of curves.

1. Introduction

Moduli spaces of stable vector bundles on projective surfaces have been the subject of a great deal of research in recent years, and much is now known about them. In contrast, moduli spaces of bundles on higher-dimensional varieties are still rather poorly understood. In this paper we extend the theory of Fourier-Mukai transforms, a useful tool in the study of moduli spaces of bundles on surfaces, to an interesting class of higher-dimensional varieties, namely those with K3 or elliptic fibrations.

1.1. Let $\pi\colon X \to S$ be a Calabi-Yau fibration, that is a connected morphism of non-singular projective varieties whose general fibre has trivial canonical bundle. Suppose that π has relative dimension one or two, so that the generic fibre is an abelian surface, a K3 surface or an elliptic curve. We shall also fix some polarisation ℓ of X.

Let Y be a component of the relative moduli space of stable sheaves on the fibration π. Points of Y represent stable sheaves supported on the fibres of π, and there is a natural map $\hat{\pi}\colon Y \to S$ sending a sheaf supported on the fibre $\pi^{-1}(s)$ to the corresponding point $s \in S$.

These relative moduli spaces have been studied by other authors, notably R. Thomas [18]. We shall denote them by $\mathcal{M}^{\ell}(X/S)$. Here we shall be mainly interested in cases when Y has the same dimension as X, and is fine, in that there is a universal sheaf on $Y \times X$. We shall show that Y is then a non-singular projective variety and the morphism $\hat{\pi}\colon Y \to S$ is another Calabi-Yau fibration, which we refer to as a Mukai dual of the fibration π.

Clearly, one would expect the geometry of the two spaces X and Y to be closely related, and indeed, this turns out to be the case. Thus, the Hodge numbers of the two spaces often coincide, and at a deeper level, one finds that certain moduli spaces of sheaves on X and Y can be identified. One of the aims of this paper is to study this geometrical relationship in more detail. It turns out that a good way to do this is to observe that the universal sheaf on $Y \times X$ induces a natural equivalence between the derived categories of coherent sheaves on the two spaces. Such equivalences are called Fourier-Mukai (FM) transforms, the first example being due to Mukai [12].

The construction outlined above was applied to surfaces fibred by elliptic curves in [4]. The resulting FM transforms were used to show that the general component of the moduli space of stable sheaves on an elliptic surface is birational to a Hilbert schemes of points. The same transforms were also used to describe a type of mirror symmetry for string theories compactified on elliptically fibred K3 surfaces [3].

In this paper we extend these techniques to higher dimensions by generalising the results of [4] to varieties fibred by K3 or abelian surfaces, or by elliptic curves, over bases of arbitrary dimension. We give some applications of the resulting FM transforms to moduli spaces of stable bundles on threefolds. One might also expect these transforms to play a rôle in the mathematical description of mirror symmetry for hyperkähler fourfolds.

Note that Calabi-Yau fibrations can have singular fibres, which in general are both non-reduced and reducible. Describing the possible forms for such fibres is a difficult problem. An interesting aspect of our methods is that we do not need to know this information. We prove a kind of 'removable singularities' result, which, put roughly, states that providing the singular fibres have sufficiently high codimension it is enough to understand the case of non-singular fibres.

1.2. We shall only consider fibrations which satisfy the following minimality condition.

Definition 1.1. *In this paper a Calabi-Yau fibration is a morphism of non-singular projective varieties $\pi\colon X \to S$ whose general fibre is a variety with trivial canonical bundle and such that $K_X \cdot C = 0$ for any curve $C \subset X$ contained in a fibre of π.*

Note that such fibrations need not be flat, in fact, a morphism of non-singular projective varieties is flat precisely when it is equidimensional. Note also that flat Calabi-Yau fibrations of relative dimension at most two (with our definition) are of one of three types: elliptic fibrations, K3 fibrations or abelian surface fibrations. Our main result is

Theorem 1.2. *Let $\pi\colon X \to S$ be a flat Calabi-Yau fibration of relative dimension at most two. Take a polarization ℓ on X and let Y be an irreducible component of $\mathcal{M}^\ell(X/S)$. Suppose that Y is fine and of the same dimension as X and that the morphism $\hat{\pi}\colon Y \to S$ is equidimensional. Let $\mathcal{P}$ be a universal sheaf on $Y \times X$. Then Y is non-singular, $\hat{\pi}$ is a flat Calabi-Yau fibration of the same type as π, and the functor*

$$\Phi^{\mathcal{P}}_{Y\to X}(-) = \mathbf{R}\pi_{X,*}(\mathcal{P} \otimes \pi_Y^*(-)),$$

where $Y \xleftarrow{\pi_Y} Y \times X \xrightarrow{\pi_X} X$ are the projection maps, is an equivalence of derived categories $\mathrm{D}(Y) \to \mathrm{D}(X)$.

There are well-known methods for finding fine components of $\mathcal{M}^\ell(X/S)$ of the right dimension which we describe in Section 7.1 below. However, the condition that $\hat{\pi}$ is equidimensional is somewhat unsatisfactory, since it is difficult to check in practice. The problem is that there conceivably exist Calabi-Yau fibrations with certain fibres X_s whose singularities are so severe that the corresponding moduli space Y_s has dimension larger than that of X_s. Nonetheless the authors do not know of an example when this problem actually occurs, and in particular, we shall show that when X has dimension at most three, and the morphism $\hat{\pi}$ is surjective, the condition that $\hat{\pi}$ is equidimensional is automatic.

The main difficulty in the proof of Theorem 1.2 is in showing that the dual variety Y is non-singular. This point did not arise in [4], because the dimension of the tangent space to Y at any point could be calculated directly using the Riemann-Roch formula. As we remarked above, much less is known about the geometry of moduli spaces of sheaves on higher-dimensional varieties, so the statement that Y is non-singular is surprisingly strong.

The proof we give uses a deep result in commutative algebra known as the intersection theorem. Since our argument might seem rather complicated at first sight, it may be worth noting that the methods we develop here have also been used to show that three-dimensional Gorenstein quotient singularities have crepant resolutions [7], a fact only known previously through case-by-case analysis.

1.3. The statement that two varieties X and Y have equivalent derived categories has some strong geometrical consequences. Firstly, certain topological invariants of X and Y should coincide. One expects, for example, that the Hodge numbers of the two spaces are equal. This statement should follow from Barannikov and Kontsevich's work on A^∞ deformations of derived categories [1]. In the meantime we have the following result, first proved in special cases by R. Thomas [18].

Proposition 1.3. *In the situation of Theorem 1.2, if X is a Calabi-Yau threefold, then so is Y, and then $h^{p,q}(X) = h^{p,q}(Y)$ for all p, q.*

A more important consequence of an equivalence of derived categories is that various moduli spaces of sheaves on the two varieties become identified. Thus we have

Theorem 1.4. *Let X be a threefold with a flat elliptic fibration $\pi\colon X \to S$ and let $\hat{\pi}\colon Y \to S$ be a Mukai dual fibration as in Theorem 1.2. Let $\mathcal{N}$ be a connected component of the moduli space of rank one, torsion-free sheaves on Y. Then there is a polarisation ℓ of X and a connected component $\mathcal{M}$ of the moduli space of stable torsion-free sheaves on X with respect to ℓ which is isomorphic to $\mathcal{N}$.*

This is a pretty result, but in practice one would like to be able to explicitly describe a given moduli space on X. This is a more delicate problem which we leave for future research. Here we content ourselves with an example; we show that a particular moduli space of stable bundles on a non-singular anticanonical divisor in $\mathbb{P}^2 \times \mathbb{P}^2$ is isomorphic to $\mathbb{P}^2$.

Plan of the paper. We start in Sections 2 and 3 with some preliminaries on moduli spaces of stable sheaves and Fourier-Mukai transforms respectively.

The intersection theorem is stated in Section 4 and interpreted geometrically in Section 5. These ideas are used in Section 6 to prove the 'removable singularities' result, Theorem 6.1, which, in turn, is used to prove Theorem 1.2 in Section 7. Sections 8 and 9 contain a more detailed treatment of the case of elliptically fibred threefolds and we conclude in Section 10 with the example of a non-singular anticanonical divisor in $\mathbb{P}^2 \times \mathbb{P}^2$.

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Conventions. We work throughout in the category of schemes over $\mathbb{C}$. Points of a scheme are always closed points. By a sheaf on a scheme X we mean a coherent $\mathcal{O}_X$-module. The bounded derived category of coherent sheaves on a scheme X is denoted $\mathrm{D}(X)$. The ith cohomology (resp. homology) sheaf of an object E of $\mathrm{D}(X)$ is denoted $H^i(E)$ (resp. $H_i(E)$), and an object of $\mathrm{D}(X)$ satisfying $H^i(E) = 0$ for all $i \neq 0$ will be referred to simply as a sheaf.

2. Moduli spaces of stable sheaves

The concept of stability of sheaves was introduced by Mumford in the case of bundles on curves and later extended to torsion-free sheaves on surfaces by Gieseker and Maruyama. More recently Simpson defined a general notion of stability for sheaves on arbitrary projective schemes. In this section we summarise some of Simpson's results. For proofs see [17].

2.1. Let X be a projective scheme over $\mathbb{C}$ and let ℓ be a polarisation of X. Thus ℓ is the first Chern class of an ample line bundle L on X. The Hilbert polynomial of a sheaf E on X with respect to the polarisation ℓ is the polynomial function $\mathrm{P}_E(n) = \chi(E \otimes L^{\otimes n})$. The coefficients of this polynomial are given in terms of the Chern classes of E by the Riemann-Roch theorem. In particular the degree of P_E is the dimension of the support of E. The unique rational multiple of P_E which is monic is called the normalised Hilbert polynomial of E and will be denoted $\mathcal{P}_E$. Polynomials of the same degree are ordered so that $\mathcal{P}_1 < \mathcal{P}_2$ precisely when $\mathcal{P}_1(n) < \mathcal{P}_2(n)$ for $n \gg 0$.

A sheaf E on X is said to be of pure dimension if the support of any non-zero subsheaf of E has the same dimension as the support of E. For example, if X is a variety and E has positive rank this is just the condition that E is torsion-free. A sheaf E is stable with respect to the polarisation ℓ if E has pure dimension, and if for all proper subsheaves $A \subset E$ one has $\mathcal{P}_A < \mathcal{P}_E$. The usual properties of stability carry over to this more general situation. For example, if A and B are stable with $\mathcal{P}_A \geq \mathcal{P}_B$ then any non-zero map $A \to B$ is an isomorphism. In particular, stable sheaves are simple.

There is a moduli space $\mathcal{M}^{\ell}(X)$ of stable sheaves on X, which in general has infinitely many components, each of which is a quasi-projective scheme which may be compactified by adding equivalence classes of semistable sheaves. An irreducible component $Y \subset \mathcal{M}^{\ell}(X)$ is said to be *fine* if Y is projective (so that there is no semistable boundary), and there exists a universal family of stable sheaves $\{\mathcal{P}_y : y \in Y\}$ on X. In particular there is a sheaf $\mathcal{P}$ on $Y \times X$, flat over Y, such that for each $y \in Y$, $\mathcal{P}_y$ is the stable sheaf on X represented by the point y. Note that by the universal property, the family $\{\mathcal{P}_y : y \in Y\}$ is complete, that is for each point $y \in Y$ the Kodaira-Spencer map $\mathrm{T}_y Y \to \mathrm{Ext}^1_X(\mathcal{P}_y, \mathcal{P}_y)$ is an isomorphism.

Any given irreducible component $Y \subset \mathcal{M}^{\ell}(X)$ parameterises sheaves E with fixed numerical invariants, and hence fixed Hilbert polynomial P. A result of Mukai [14, Theorem A.6] implies that Y is fine whenever the integers $\chi(E \otimes L^{\otimes n})$ have no common factor.

We now turn to relative moduli spaces. Let $\pi\colon X \to S$ be a morphism of projective schemes, and fix a polarization ℓ of X. For each point $s \in S$, ℓ induces a polarization ℓ_s of the fibre X_s of π, defined by restricting an ample line bundle L representing ℓ to the subscheme X_s. In this situation there is a relative moduli scheme

$$\hat{\pi} : \mathcal{M}^{\ell}(X/S) \longrightarrow S$$

whose fibre over a point $s \in S$ is naturally the moduli scheme $\mathcal{M}^{\ell_s}(X_s)$ of stable sheaves on the fibre X_s, with respect to the polarisation ℓ_s. Points of $\mathcal{M}^{\ell}(X/S)$ correspond to stable sheaves E on X whose scheme-theoretic support is contained in some fibre X_s of π. The morphism $\hat{\pi}$ takes the point representing such a sheaf E to the corresponding point $s \in S$. The existence of such moduli schemes is due originally to Simpson (see [17] or [18, Section 4]) but is now standard.

As before, if Y is a connected component of $\mathcal{M}^{\ell}(X/S)$ parameterising sheaves E for which the integers $\chi(E \otimes L^{\otimes n})$ have no common factor, then Y is fine, in that Y is projective and there is a universal sheaf $\mathcal{P}$ on $Y \times X$. This universal sheaf is actually supported on the closed subscheme $j\colon Y \times_S X \hookrightarrow Y \times X$. Thus there is a sheaf $\mathcal{P}^+$ on $Y \times_S X$, flat over Y, such that $\mathcal{P} = j_* \mathcal{P}^+$. For each point $y \in Y$ with $\hat{\pi}(y) = s$, one has $\mathcal{P}_y = i_{y*} \mathcal{P}^+_y$, where $\mathcal{P}^+_y$ is the restriction of $\mathcal{P}^+$ to the subscheme $i_y\colon \{y\} \times X_s \hookrightarrow X$.

2.2. We finish this section with two simple results about stable sheaves. Both are presumably well-known, but we include proofs since we were unable to find a suitable reference.

Lemma 2.1. *Let $\pi\colon X \to S$ be a morphism of non-singular projective varieties of relative dimension one. Let ℓ be a polarisation of X and let ϕ be the pull-back of a polarisation from S. Let E be a torsion-free sheaf on X whose restriction to the general fibre of π is stable. Then there exists an integer m_0 such that for all $m \geq m_0$ the sheaf E is stable with respect to the polarisation $\ell + m\phi$.*

Proof. Let d be the dimension of S. For each proper subsheaf $A \subset E$ put

$$\Delta(A) = r(A)\,\mathrm{c}_1(E) - r(E)\,\mathrm{c}_1(A) \in H^2(X,\mathbb{Z}).$$

The sheaf E is called μ-stable with respect to a polarisation ω if for all such A one has $\Delta(A)\cdot\omega^d > 0$. In particular μ-stable sheaves are stable. The lemma follows from the following two facts. Firstly, for fixed m, the numbers $\Delta(A)\cdot(\ell + m\phi)^d$ are bounded below. Secondly, the fact that the restriction of E to the general fibre of π is stable implies that $\Delta(A)\cdot\phi^d > 0$. □

Lemma 2.2. *Let X be a non-singular, simply-connected projective variety. Let $\mathcal{N}$ denote the union of those connected components of* $\mathrm{Hilb}(X)$ *which parametrise closed subschemes of codimension greater than 1. Let $\mathcal{M}$ be the moduli space of (stable) torsion-free sheaves with rank one and zero first Chern class. Then there is a morphism of schemes $\alpha\colon \mathcal{N} \to \mathcal{M}$ inducing a bijection on closed points, which sends a subscheme $C \subset X$ to the ideal sheaf $\mathcal{I}_C$.*

Proof. There is a universal sheaf $\mathcal{P}$ on $\mathcal{N}\times X$, flat over $\mathcal{N}$, such that for each point $s \in \mathcal{N}$, $\mathcal{P}_s$ is the ideal sheaf of the corresponding subscheme $C \subset X$. The existence of the map α follows from the universal property of $\mathcal{M}$.

A point of $\mathcal{M}$ corresponds to a torsion-free sheaf E of rank 1 and zero first Chern class. There is a short exact sequence

$$0 \longrightarrow E \longrightarrow E^{**} \longrightarrow T \longrightarrow 0$$

where T is supported in codimension > 1 and the double dual sheaf E^{**} is reflexive of rank 1, hence invertible [16, Lemma II.1.1.15]. Since X is simply-connected and $\mathrm{c}_1(E^{**}) = 0$, one has $E^{**} = \mathcal{O}_X$, so $E = \mathcal{I}_C$ for some subscheme $C \in \mathcal{N}$. Thus α is bijective on points. □

The authors do not know of any examples where the map α of the Lemma fails to be an isomorphism.

3. Fourier-Mukai transforms

Let X be a non-singular projective variety. The derived category of X is a triangulated category whose objects are complexes of sheaves on X with bounded and coherent homology sheaves. This category contains a large amount of geometrical information about X. Indeed, if $\pm K_X$ is ample, Bondal and Orlov [2] show that $\mathrm{D}(X)$ considered up to triangle-preserving equivalence completely determines X. In general however, there exist pairs of non-singular projective varieties (X, Y) for which there are triangle-preserving equivalences $\Phi\colon \mathrm{D}(Y) \to \mathrm{D}(X)$. Such equivalences are called Fourier-Mukai transforms. In this section we shall describe some of the geometrical consequences of a pair of varieties being related in this way. First however, we address the problem of constructing examples of FM transforms.

3.1. Constructing FM transforms. A theorem of Orlov [15] states that for any FM transform $\Phi\colon \mathrm{D}(Y) \to \mathrm{D}(X)$, there is an object $\mathcal{P}$ of $\mathrm{D}(Y\times X)$ such that Φ is isomorphic to the functor $\Phi^{\mathcal{P}}_{Y\to X}$ given by the formula

$$\Phi^{\mathcal{P}}_{Y\to X}(-) = \mathbf{R}\pi_{X,*}(\mathcal{P} \overset{\mathbf{L}}{\otimes} \pi_Y^*(-)),$$

where $Y \xleftarrow{\pi_Y} Y \times X \xrightarrow{\pi_X} X$ are the projection maps. It remains to determine which objects $\mathcal{P}$ give equivalences. Suppose for simplicity that $\mathcal{P}$ is a sheaf on $Y \times X$, flat over Y and put $\Phi = \Phi^{\mathcal{P}}_{Y \to X}$.

If A and B are objects of $\mathrm{D}(X)$ we put

$$\mathrm{Hom}^i_{\mathrm{D}(X)}(A, B) = \mathrm{Hom}_{\mathrm{D}(X)}(A, B[i])$$

where $[i]$ is the functor which shifts all complexes of $\mathrm{D}(X)$ to the left by i places. If A and B are sheaves on X, that is complexes whose homology is concentrated in degree zero, one has $\mathrm{Hom}^i_{\mathrm{D}(X)}(A, B) = \mathrm{Ext}^i_X(A, B)$.

The key observation is that if Φ is a FM transform then

$$\mathrm{Hom}^i_{\mathrm{D}(X)}(\Phi(A), \Phi(B)) = \mathrm{Hom}^i_{\mathrm{D}(Y)}(A, B).$$

This suggests the idea of thinking of $\mathrm{Hom}^\bullet(-,-)$ as analogous to an inner product, so that the functor Φ becomes an isometry. Pursuing the analogy, note that for distinct points y_1, y_2 of Y, one has $\mathrm{Ext}^i_Y(\mathcal{O}_{y_1}, \mathcal{O}_{y_2}) = 0$ for all i. In this way (see also Lemma 5.3) the set of sheaves $\{\mathcal{O}_y : y \in Y\}$ can be thought of as an orthogonal basis for $\mathrm{D}(Y)$. Applying Φ must give an orthogonal basis of sheaves $\{\mathcal{P}_y : y \in Y\}$ on X, parameterised by Y. The point of the theorem below is that conversely, such families gives rise to FM transforms.

Theorem 3.1. *Let X and Y be non-singular projective varieties of the same dimension, and $\mathcal{P}$ a sheaf on $Y \times X$, flat over Y. Then the functor*

$$\Phi^{\mathcal{P}}_{Y \to X} \colon \mathrm{D}(Y) \longrightarrow \mathrm{D}(X)$$

is an equivalence of categories if and only if, for any point $y \in Y$,

$$\mathrm{End}_X(\mathcal{P}_y) = \mathbb{C} \text{ and } \mathcal{P}_y \otimes \omega_X = \mathcal{P}_y,$$

and for any pair of distinct points y_1, y_2 of Y, and any integer i,

$$\mathrm{Ext}^i_X(\mathcal{P}_{y_1}, \mathcal{P}_{y_2}) = 0.$$

Proof. A complete proof, based on ideas of Bondal, Orlov and Mukai appears in [5]. There is also a more general version, which we shall not need in this paper, in which $\mathcal{P}$ is allowed to be an arbitrary object of $\mathrm{D}(Y \times X)$. $\square$

3.2. Applications. Suppose $\Phi = \Phi^{\mathcal{P}}_{Y \to X} \colon \mathrm{D}(Y) \to \mathrm{D}(X)$ is a FM transform. The inverse of Φ is the functor $\Psi = \Phi^{\mathcal{Q}}_{X \to Y} \colon \mathrm{D}(X) \to \mathrm{D}(Y)$ where $\mathcal{Q}$ is the object

$$\mathcal{Q} = \mathbf{R}\mathcal{H}om_{\mathcal{O}_{Y \times X}}(\mathcal{P}, \pi_X^* \omega_X)[n],$$

and n is the dimension of X and Y. Thus there are isomorphisms

$$\Psi \circ \Phi \cong \mathrm{id}_{\mathrm{D}(Y)}, \qquad \Phi \circ \Psi \cong \mathrm{id}_{\mathrm{D}(X)}.$$

Using the Chern classes of the objects $\mathcal{P}$ and $\mathcal{Q}$ one can define a pair of mutually inverse correspondences relating the cohomology of X and Y. This leads to the following result. For a proof see [18, Theorem 4.23] or [13, Theorem 4.9].

Proposition 3.2. *Let X and Y be non-singular projective varieties. A FM transform $\Phi\colon \mathrm{D}(Y) \to \mathrm{D}(X)$ induces isomorphisms*

$$\bigoplus_i H^{i,i+k}(Y) \longrightarrow \bigoplus_i H^{i,i+k}(X)$$

for each integer k. □

Proposition 1.3 follows easily from Proposition 3.2 once one observes that Serre duality requires that if $\Phi\colon \mathrm{D}(Y) \to \mathrm{D}(X)$ is a FM transform then Y has trivial canonical bundle precisely when X does [6, Lemma 2.1].

The second application of FM transforms we wish to consider is to moduli spaces of sheaves. Before explaining the general method we should introduce some notation. Let $\Phi\colon \mathrm{D}(Y) \to \mathrm{D}(X)$ be a FM transform. Given a sheaf E on Y we write $\Phi^i(E)$ for the ith cohomology sheaf of the object $\Phi(E)$ of $\mathrm{D}(X)$.

Definition 3.3. *A sheaf E on Y is said to be Φ-WIT$_i$ if $\Phi^j(E) = 0$ unless $j = i$. A sheaf E is Φ-WIT if it is Φ-WIT$_i$ for some i. One then refers to the sheaf $\hat{E} = \Phi^i(E)$ as the transform of E.*

The reason that FM transforms can be used to compute moduli spaces of sheaves is that they preserve families [5, Proposition 4.2]. Thus if $\{\mathcal{E}_s : s \in S\}$ is a family of Φ-WIT sheaves on Y, then $\{\hat{\mathcal{E}}_s : s \in S\}$ is a family of sheaves on X. As a consequence one has the well-known

Proposition 3.4. *Let $\{\mathcal{E}_s : s \in S\}$ be a complete family of pairwise non-isomorphic sheaves on Y, over a connected, projective scheme S. Suppose that for each point $s \in S$ the sheaf $\mathcal{E}_s$ is Φ-WIT and the transform $\hat{\mathcal{E}}_s$ is stable with respect to some polarisation ℓ of X. Then the natural map $S \to \mathcal{M}^\ell(X)$ which sends a point $s \in S$ to the point representing the sheaf $\hat{\mathcal{E}}_s$ is an isomorphism onto a connected component of $\mathcal{M}^\ell(X)$.*

Proof. The map exists by the universal property of $\mathcal{M}^\ell(X)$ and the fact that $\{\hat{\mathcal{E}}_s : s \in S\}$ is a family of stable sheaves on X. Since this family is complete this map induces isomorphisms on tangent spaces. It is injective on points because the sheaves $\hat{\mathcal{E}}_s$ are pairwise non-isomorphic. The result follows. □

In practice one often takes the sheaves $\mathcal{E}_s$ to be torsion-free of rank 1, since moduli spaces of such sheaves can be related to Hilbert schemes via Lemma 2.2. Many moduli spaces have been computed in this way.

4. The Intersection Theorem

Our proof of Theorem 1.2 uses a celebrated result of commutative algebra called the intersection theorem. The aim of this section is to state and prove a strong version of this theorem using Hochster's results on the existence of maximal Cohen-Macaulay modules. Our main reference is [10].

We start by defining the depth of an arbitrary (not necessarily finitely generated) module over a local ring.

Definition 4.1. *Let $(A, \mathfrak{m})$ be a Noetherian local ring. The depth of an A-module M is defined to be*

$$\operatorname{depth}_A(M) = \inf\{i \in \mathbb{Z} : \operatorname{Ext}^i_A(A/\mathfrak{m}, M) \neq 0\}.$$

In general modules may have infinite depth, but for finitely generated modules the above definition is always finite and agrees with the usual definition of depth in terms of A-sequences. See [10, Theorem 1.6].

A fairly immediate consequence of this definition is the following acyclicity lemma of Peskine and Szpiro. For a proof see [10, Lemma 1.3].

Lemma 4.2. *Let $(A, \mathfrak{m})$ be a Noetherian local ring and*

$$0 \longrightarrow M_s \longrightarrow M_{s-1} \longrightarrow \cdots \longrightarrow M_0 \longrightarrow 0,$$

a finite complex of A-modules such that each non-zero homology module $H_i(M_\bullet)$ has depth 0. Then

$$\operatorname{depth}_A(M_i) \geq i \qquad 0 \leq i \leq s$$

implies that $H_i(M_\bullet) = 0$ for all $i > 0$, and

$$\operatorname{depth}_A(M_i) > i \qquad 0 \leq i \leq s$$

implies that $M_\bullet$ is exact, that is $H_i(M_\bullet) = 0$ for all i. $\square$

We can now state the main result of this section. The first part is called the intersection theorem; we reproduce the proof for the reader's convenience. The second part is a slight strengthening which we prove using similar methods. See [10, Theorem 1.13].

Theorem 4.3. *Let $(A, \mathfrak{m})$ be a Noetherian local $\mathbb{C}$-algebra of dimension d and let*

$$0 \longrightarrow M_s \longrightarrow M_{s-1} \longrightarrow \cdots \longrightarrow M_0 \longrightarrow 0,$$

be a non-exact complex of finitely-generated free A-modules such that each homology module $H_i(M_\bullet)$ is a finite A-module. Then $s \geq d$. Furthermore, if $s = d$ and $H_0(M_\bullet) \cong A/\mathfrak{m}$, then

$$H_i(M_\bullet) = 0 \text{ for all } i \neq 0,$$

and A is regular.

Proof. We may assume that $(A, \mathfrak{m})$ is complete. Fixing a system of parameters $(x_1, \cdots, x_d)$ for A, let R denote the complete regular local ring

$$R = \mathbb{C}[[x_1, \cdots x_d]],$$

and let $\mathfrak{n}$ denote its maximal ideal. Note that A is finite as an R-module.

Work of Griffith and Hochster [10, Theorem 1.8] shows that in this situation there is a (not-necessarily finitely-generated) A-module C such that C is free as an R-module. It follows immediately that $\operatorname{depth}_A(C) = d$. One says that C is a balanced big Cohen-Macaulay module for A.

Let us put

$$N = A \otimes_R (R/\mathfrak{n}).$$

Then N is a finite A-module satisfying

$$\operatorname{Tor}_p^A(N, C) = 0 \text{ for all } p > 0, \tag{1}$$

and $N \otimes C \neq 0$. It follows from this that for any non-zero finite length A-module P, the module $P \otimes C$ is non-zero.

Suppose now that $s < d$, and tensor the complex $M_\bullet$ by C. Each term of the resulting complex $M_\bullet \otimes C$ has depth d because it is a direct sum of finitely many copies of C. Applying the acyclicity lemma then shows that $M_\bullet \otimes C$ is exact. Consider the third-quadrant spectral sequence

$$E_2^{p,q} = \operatorname{Tor}^A_{-p}(H_{-q}(M_\bullet), C) \implies H_{-(p+q)}(M_\bullet \otimes C).$$

If the complex $M_\bullet$ is not exact, let q_0 be the least integer such that $H_q(M_\bullet)$ is non-zero. Then the term $E_2^{0,-q_0}$ of the sequence is non-zero and survives to infinity, which is a contradiction. This proves the first part.

Assume now that $s = d$. By the acyclicity lemma again, the complex $M_\bullet \otimes C$ is exact except at the right-hand end, so the spectral sequence $E_2^{p,q}$ converges to 0 unless $p + q = 0$. In particular we must have $E_2^{-1,0} = 0$, so since we have assumed $H_0(M_\bullet) = A/\mathfrak{m}$, we have

$$\operatorname{Tor}^A_1(A/\mathfrak{m}, C) = 0.$$

Using the fact that the finite A-module N has a composition series whose factors are isomorphic to $A/\mathfrak{m}$, and (1) above, it is easy to see that for any finite A-module P

$$\operatorname{Tor}^A_p(P, C) = 0 \text{ for all } p > 0.$$

Thus the spectral sequence degenerates and we can conclude that $H_i(M_\bullet) = 0$ for $i > 0$, and so $M_\bullet$ is a finite free resolution of $A/\mathfrak{m}$. It follows that A is regular. □

5. Support and homological dimension

In this section we translate the intersection theorem into geometrical language whence it becomes a statement relating the homological dimension of a complex of sheaves to the codimension of the support of its homology sheaves. Throughout X denotes an arbitrary scheme of finite type over $\mathbb{C}$.

Definition 5.1. *The* support *of an object E of* $\mathrm{D}(X)$*, written* $\operatorname{supp}(E)$*, is the union of the supports of the homology sheaves $H_i(E)$ of E. It is a closed subset of X.*

Definition 5.2. *Given a non-zero object E of* $\mathrm{D}(X)$*, the* homological dimension *of E, written* $\operatorname{hd}(E)$*, is equal to the smallest integer s such that E is quasi-isomorphic to a complex of locally free $\mathcal{O}_X$-modules of length s. If no such integer exists we put* $\operatorname{hd}(E) = \infty$.

The following two simple results allow one to calculate both the support and homological dimension of a given object E of $\mathrm{D}(X)$ from a knowledge of the vector spaces

$$H_i(E \overset{\mathbf{L}}{\otimes} \mathcal{O}_x) = \operatorname{Hom}^i_{\mathrm{D}(X)}(E, \mathcal{O}_x)^\vee, \tag{2}$$

as x ranges over the points of X. To check that the above two spaces really are equal, let $f : \{x\} \hookrightarrow X$ be the inclusion map, and apply [11, Proposition II.5.6, Corollary II.5.11].

Lemma 5.3. *Let E be an object of* $\mathrm{D}(X)$*, and fix a point $x \in X$. Then*

$$x \in \operatorname{supp}(E) \iff \exists i \in \mathbb{Z} : \operatorname{Hom}^i_{\mathrm{D}(X)}(E, \mathcal{O}_x) \neq 0.$$

Proof. There is a spectral sequence

$$E_2^{p,q} = \mathrm{Ext}^p_X(H_q(E), \mathcal{O}_x) \implies \mathrm{Hom}^{p+q}_{\mathrm{D}(X)}(E, \mathcal{O}_x).$$

If x lies in the support of E, let q_0 be the minimal value of q such that x is contained in the support of the homology sheaf $H_q(E)$. Then there is a non-zero element of E_2^{0,q_0} which survives to give a non-zero element of $\mathrm{Hom}^{q_0}_{\mathrm{D}(X)}(E, \mathcal{O}_x)$. The converse is clear. □

Proposition 5.4. *Let X be a quasi-projective scheme, take a non-zero object E of $\mathrm{D}(X)$ and let $s \geq 0$ be an integer such that for all points $x \in X$,*

$$H_i(E \overset{\mathbf{L}}{\otimes} \mathcal{O}_x) = 0 \text{ unless } 0 \leq i \leq s.$$

Then E is quasi-isomorphic to a complex of locally free sheaves of the form

$$0 \longrightarrow L_s \longrightarrow L_{s-1} \longrightarrow \cdots \longrightarrow L_0 \longrightarrow 0.$$

In particular, E has homological dimension at most s.

Proof. If $s = 0$ the result follows from [4, Lemma 4.3], so assume s positive. Using (2) and applying the argument used to prove the last lemma, one sees that $H_i(E) = 0$ for all $i < 0$. Since X is quasi-projective, every coherent sheaf on X is the quotient of a finite rank, locally free sheaf, so the dual of [11, Lemma I.4.6] implies that E is quasi-isomorphic to a complex of locally free sheaves on X of the form

$$\cdots \longrightarrow L_i \xrightarrow{d_i} L_{i-1} \xrightarrow{d_{i-1}} L_{i-2} \longrightarrow \cdots \xrightarrow{d_1} L_0 \longrightarrow 0.$$

Consider the object F of $\mathrm{D}(X)$ defined by the truncated complex

$$\cdots \longrightarrow L_{s+3} \xrightarrow{d_{s+3}} L_{s+2} \xrightarrow{d_{s+2}} L_{s+1} \xrightarrow{d_{s+1}} L_s \longrightarrow 0.$$

Applying the functor $-\otimes \mathcal{O}_x$ and comparing with E, one sees that for any $x \in X$, $H_i(F \overset{\mathbf{L}}{\otimes} \mathcal{O}_x) = 0$ for $i > s$ and $i < s$. Applying the $s = 0$ case, this implies that F has homological dimension 0. It follows that $H_i(E) = 0$ for all $i > s$, and that $L_s/\operatorname{im}(d_{s+1})$ is locally free.

Consider the short exact sequence of complexes

$$0 \longrightarrow A \longrightarrow L \longrightarrow B \longrightarrow 0,$$

where A is the complex

$$\cdots \longrightarrow L_{s+2} \xrightarrow{d_{s+2}} L_{s+1} \longrightarrow \operatorname{im}(d_{s+1}) \longrightarrow 0,$$

and B is the length s complex of locally free sheaves

$$0 \longrightarrow L_s/\operatorname{im}(d_{s+1}) \longrightarrow L_{s-1} \xrightarrow{d_{s-1}} \cdots \longrightarrow L_0 \longrightarrow 0.$$

It is enough to show that A is quasi-isomorphic to zero. But this is clear since $H_i(A) = H_i(L) = 0$ for $i > s$. □

In geometrical language the intersection theorem becomes

Corollary 5.5. *Let X be a scheme of finite type over $\mathbb{C}$ and E a non-trivial object of $\mathrm{D}(X)$. Then for any irreducible component Γ of $\operatorname{supp}(E)$ one has an inequality*

$$\operatorname{codim}(\Gamma) \leq \operatorname{hd}(E).$$

Proof. We may assume that E is a complex of locally free sheaves of finite length $s \geq 0$. Let E_0 be the restriction of E to the affine subscheme $\operatorname{Spec}(A)$ of X, where A is the local $\mathbb{C}$-algebra $\mathcal{O}_{X,\Gamma}$. Then E_0 is non-trivial, and each homology sheaf $H_i(E_0)$ is a finite A-module [8, Corollary 2.18]. Furthermore, the dimension of A is equal to the codimension of Γ. Theorem 4.3 now gives the result. □

The second part of Theorem 4.3, together with Lemma 5.3 and Proposition 5.4 gives

Corollary 5.6. *Let X be an irreducible quasi-projective scheme of dimension n over $\mathbb{C}$ and fix a point $x \in X$. Suppose that there is an object E of $\mathrm{D}(X)$ such that for any point $z \in X$, and any integer i,*

$$\operatorname{Hom}^i_{\mathrm{D}(X)}(E, \mathcal{O}_z) = 0 \text{ unless } z = x \text{ and } 0 \leq i \leq n.$$

Suppose also that $H_0(E) \cong \mathcal{O}_x$. Then X is non-singular at x, and $E \cong \mathcal{O}_x$. □

6. A removable singularities result

In this section we use the results of the last two sections to prove the following strengthening of Theorem 3.1. This is the result which allows us to deal with singular fibres of Calabi-Yau fibrations.

Theorem 6.1. *Let X be a non-singular projective variety of dimension n and let $\{\mathcal{P}_y : y \in Y\}$ be a complete family of simple sheaves on X parameterised by an irreducible projective scheme Y of dimension n. Suppose that*

$$\operatorname{Hom}_X(\mathcal{P}_{y_1}, \mathcal{P}_{y_2}) = 0$$

for any distinct points $y_1, y_2 \in Y$ and that the closed subscheme

$$\Gamma(\mathcal{P}) = \{(y_1, y_2) \in Y \times Y : \operatorname{Ext}^i_X(\mathcal{P}_{y_1}, \mathcal{P}_{y_2}) \neq 0 \text{ for some } i \in \mathbb{Z}\}$$

of $Y \times Y$ has dimension at most $n+1$. Suppose also that $\mathcal{P}_y \otimes \omega_X \cong \mathcal{P}_y$ for all $y \in Y$. Then Y is a non-singular variety and the functor

$$\Phi^{\mathcal{P}}_{Y \to X} \colon \mathrm{D}(Y) \longrightarrow \mathrm{D}(X)$$

is an equivalence of categories.

Proof. The sheaf $\mathcal{P}$ is flat over Y, and X is non-singular, so given a point $(y, x) \in Y \times X$, the complex

$$\mathcal{P} \overset{\mathbf{L}}{\otimes} \mathcal{O}_{(y,x)} \cong \mathcal{P}_y \overset{\mathbf{L}}{\otimes} \mathcal{O}_x$$

has bounded homology. It follows from Proposition 5.4 that $\mathcal{P}$ has finite homological dimension, and in particular, the object

$$\mathcal{P}^\vee = \mathbf{R}\mathcal{H}om_{\mathcal{O}_{Y\times X}}(\mathcal{P}, \mathcal{O}_{Y\times X})$$

has bounded homology. By Grothendieck-Verdier duality the functor $\Phi = \Phi^{\mathcal{P}}_{Y \to X}$ has a left adjoint

$$\Psi \colon \mathrm{D}(X) \longrightarrow \mathrm{D}(Y).$$

By composition of correspondences [12, Proposition 1.3], there is an object $\mathcal{S}$ of $\mathrm{D}(Y \times Y)$ such that there is an isomorphism of functors

$$\Psi \circ \Phi(-) \cong \mathbf{R}\pi_{2,*}(\mathcal{S} \overset{\mathbf{L}}{\otimes} \pi_1^*(-)),$$

where π_1 and π_2 are the projections of the product $Y \times Y$ onto its two factors.

For any point $y \in Y$, the derived restriction of the object $\mathcal{S}$ to the subscheme $\{y\} \times Y$ is just $\Psi\Phi\mathcal{O}_y$. It follows that for any pair of points (y_1, y_2) in $Y \times Y$,

$$H_p(\mathcal{S} \overset{\mathbf{L}}{\otimes} \mathcal{O}_{(y_1,y_2)})^\vee = \mathrm{Hom}^p_{\mathrm{D}(Y)}(\Psi\Phi\mathcal{O}_{y_1}, \mathcal{O}_{y_2}) = \mathrm{Ext}^p_X(\mathcal{P}_{y_1}, \mathcal{P}_{y_2}) \tag{3}$$

using the adjunction $\Psi \vdash \Phi$. Since X is non-singular these groups vanish unless $0 \leq p \leq n$.

If $y_1 \neq y_2$ are distinct points of Y,

$$\mathrm{Ext}^n_X(\mathcal{P}_{y_2}, \mathcal{P}_{y_1}) = \mathrm{Hom}_X(\mathcal{P}_{y_1}, \mathcal{P}_{y_2}) = 0,$$

so if we define $\mathcal{E}$ to be the restriction of $\mathcal{S}$ to the complement of the diagonal in $Y \times Y$, Proposition 5.4 implies that $\mathcal{E}$ has homological dimension $n-2$. By hypothesis the support of $\mathcal{E}$ has codimension at least $n-1$, so applying Proposition 5.5 shows that $\mathcal{E} \cong 0$. It follows that the support of $\mathcal{S}$ is the diagonal, and hence the groups (3) vanish unless $y_1 = y_2$.

Fix a point $y \in Y$. For any other point $z \in Y$,

$$\mathrm{Hom}^i_{\mathrm{D}(Y)}(\Psi\Phi\mathcal{O}_y, \mathcal{O}_z) = \mathrm{Ext}^i_X(\mathcal{P}_y, \mathcal{P}_z),$$

and these groups vanish unless $y = z$ and $0 \leq i \leq n$. We claim that

$$H_0(\Psi\Phi\mathcal{O}_y) = \mathcal{O}_y.$$

Assuming this for the moment, note that Corollary 5.6 now implies that Y is non-singular, and so Theorem 6.1 follows from Theorem 3.1 (or from a simple piece of category theory [5, Theorem 2.3]).

To prove the claim note first that there is a unique map $\Psi\Phi\mathcal{O}_y \to \mathcal{O}_y$, so we obtain a triangle

$$C \longrightarrow \Psi\Phi\mathcal{O}_y \longrightarrow \mathcal{O}_y \longrightarrow C[1] \tag{4}$$

for some object C of $\mathrm{D}(Y)$, which must be supported at y. Applying the functor $\mathrm{Hom}_{\mathrm{D}(Y)}(-, \mathcal{O}_y)$ and using the adjunction $\Psi \vdash \Phi$ gives a long exact sequence

$$0 \longrightarrow \mathrm{Hom}^0_{\mathrm{D}(Y)}(\mathcal{O}_y, \mathcal{O}_y) \longrightarrow \mathrm{Hom}^0_{\mathrm{D}(X)}(\Phi\mathcal{O}_y, \Phi\mathcal{O}_y) \longrightarrow \mathrm{Hom}^0_{\mathrm{D}(Y)}(C, \mathcal{O}_y)$$

$$\longrightarrow \mathrm{Hom}^1_{\mathrm{D}(Y)}(\mathcal{O}_y, \mathcal{O}_y) \xrightarrow{\epsilon} \mathrm{Hom}^1_{\mathrm{D}(X)}(\Phi\mathcal{O}_y, \Phi\mathcal{O}_y) \longrightarrow \cdots$$

The homomorphism ϵ is just the Kodaira-Spencer map for the family $\{\mathcal{P}_y : y \in Y\}$, [5, Lemma 4.4], which is an isomorphism by assumption. Since $\Phi\mathcal{O}_y$ is simple, it follows that

$$\mathrm{Hom}^i_{\mathrm{D}(Y)}(C, \mathcal{O}_y) = 0 \text{ for all } i \leq 0,$$

and so, by the argument of Lemma 5.3, $H_i(C) = 0$ for all $i \leq 0$. Taking homology of the triangle (4) shows that $H_0(\Psi\Phi\mathcal{O}_y) = \mathcal{O}_y$ as claimed. □

Note that the condition $\mathrm{Hom}_X(\mathcal{P}_{y_1}, \mathcal{P}_{y_2}) = 0$ for all distinct y_1 and y_2 is automatic for *stable* moduli spaces which we will be considering in what follows.

7. FM transforms for Calabi-Yau fibrations

In this section we use Theorem 6.1 to prove our main result, Theorem 1.2. First we explain how to construct components of the moduli space of stable sheaves on a Calabi-Yau fibration which satisfy the hypotheses of the theorem. The reader should compare the treatment given by Thomas [18, Section 4].

7.1. Let $\pi\colon X \to S$ be a Calabi-Yau fibration, fix a polarisation ℓ of X and let X_s be a non-singular fibre of π with its induced polarisation ℓ_s. Let E be a sheaf on X_s, stable with respect to ℓ_s, and let Y be the irreducible component of $\mathcal{M}^\ell(X/S)$ containing a point representing the sheaf E.

Suppose first that π has relative dimension two. By the Riemann-Roch theorem on X_s, the Hilbert polynomial of E with respect to ℓ is

$$\mathrm{P}_E(n) = \chi(E) + \mathrm{c}_1(E) \cdot n\ell_s + \frac{1}{2}\,\mathrm{r}(E)(n\ell_s)^2.$$

If the integers $\mathrm{P}_E(n)$ have no common factor then the moduli space Y is fine.

There is an open subset $U \subset S$, such that all the fibres X_s over points $s \in U$ are non-singular. The fibre Y_s of $\hat{\pi}\colon Y \to S$ over a point $s \in U$ is a component of the moduli space of stable sheaves on the surface X_s. Results of Mukai [13, Theorems 0.1, 1.17] imply that Y_s is non-singular and either empty or of dimension

$$\mathrm{c}_1(E)^2 - 2\,\mathrm{r}(E)\,\chi(E) + \chi(\mathcal{O}_{X_s})\,\mathrm{r}(E)^2 + 2.$$

Thus the moduli space Y has the same dimension as X providing $\hat{\pi}$ is surjective and

$$2\,\mathrm{r}(E)\,\chi(E) = \chi(\mathcal{O}_{X_s})\,\mathrm{r}(E)^2 + \mathrm{c}_1(E)^2.$$

In this case the general fibre of $\hat{\pi}$ will be a Calabi-Yau surface of the same type as the general fibre of π.

The condition that $\hat{\pi}$ is surjective is satisfied providing $\mathrm{c}_1(E)$ is the restriction of the first Chern class of some line bundle on X. Note that in the case when X and Y have dimension three, this is enough to show that $\hat{\pi}$ is equidimensional, because the assumption that Y is irreducible prevents the dimension of the fibres of $\hat{\pi}$ from jumping.

Example 7.1. Suppose $\pi\colon X \to S$ is a K3 fibration, take a non-singular fibre X_s of π and let E be the ideal sheaf of a point on X_s. Then $\chi(E) = 1$ and for any polarisation ℓ of X the component $Y \subset \mathcal{M}^\ell(X/S)$ containing E is fine and has the same dimension as X. The resulting FM transforms are relative versions of Mukai's reflection functor [14, Section 2].

Suppose now that π has relative dimension one. As before, let E be a stable sheaf on a non-singular fibre X_s of π, necessarily an elliptic curve. The numerical invariants of E are simply its rank r and degree d. The condition that the irreducible component Y of $\mathcal{M}^\ell(X/S)$ containing E be fine is just that d is coprime to both r and the degree of the induced polarization ℓ_s. Whenever this happens the resulting moduli space has the same dimension as X and is elliptically fibred over S. We shall give examples later.

7.2. The main difficulty in the proof of Theorem 1.2 is dealing with singularities in the fibres of π, so we first consider the case when all the fibres of π are non-singular. To be definite we shall assume that π has relative dimension two.

Take a polarization ℓ of X and let Y be a fine component of the relative moduli scheme $\mathcal{M}^\ell(X/S)$ of the same dimension as X. Assume that $\hat{\pi}\colon Y \to S$ is flat. Each fibre of $\hat{\pi}$ is non-singular, so Y is a non-singular projective variety. Let $\mathcal{P}^+$ be a universal sheaf on $Y \times_S X$. Extending by zero we obtain a sheaf $\mathcal{P}$ on $Y \times X$. By definition $\mathcal{P}$ is flat over Y.

Fix a point $s \in S$, and let $i\colon X_s \hookrightarrow X$ be the inclusion morphism. For any point $y \in Y$ with $\hat{\pi}(y) = s$, one has $\mathcal{P}_y = i_*\mathcal{P}^+_y$. Thus to check the first condition of Theorem 3.1 it is enough to show that $i^*(\omega_X) = \mathcal{O}_{X_s}$. This is immediate from the adjunction formula, since X_s has trivial canonical bundle and trivial normal bundle.

For the second condition we may assume that the two sheaves $\mathcal{P}_{y_1}$, $\mathcal{P}_{y_2}$ are supported on the same fibre of $\pi\colon X \to S$, otherwise their Ext-groups trivially vanish. Then

$$\operatorname{Ext}^i_X(\mathcal{P}_{y_1}, \mathcal{P}_{y_1}) = \operatorname{Ext}^i_X(i_*\mathcal{P}^+_{y_1}, i_*\mathcal{P}^+_{y_2}) = \operatorname{Hom}^i_{X_s}(\mathbf{L}i^* i_*\mathcal{P}^+_{y_1}, \mathcal{P}^+_{y_2}),$$

by the adjunction $\mathbf{L}i^* \vdash i_*$. Furthermore

$$\mathbf{L}i^* i_*(\mathcal{P}^+_{y_1}) = \mathcal{P}^+_{y_1} \overset{\mathbf{L}}{\otimes} \mathbf{L}i^* i_*(\mathcal{O}_{X_s})$$

by the projection formula. Since S is non-singular, one can write down the Koszul resolution for $\mathcal{O}_s$ on S and pull-back via π to obtain a locally free resolution of $\mathcal{O}_{X_s}$ on X. This gives

$$\mathbf{L}_q i^* i_* \mathcal{P}^+_{y_1} = \mathcal{P}^+_{y_1} \otimes \bigwedge^q \mathcal{O}^{\oplus m}_{X_s},$$

for $0 \le q \le s$, where $m = \dim S$ is the dimension of the base. Now there is a spectral sequence

$$E_2^{p,q} = \operatorname{Ext}^p_{X_s}(\mathbf{L}_q i^* i_* \mathcal{P}^+_{y_1}, \mathcal{P}^+_{y_2}) \implies \operatorname{Ext}^{p+q}_X(\mathcal{P}_{y_1}, \mathcal{P}_{y_2}),$$

so it is enough to know that

$$\operatorname{Ext}^i_{X_s}(\mathcal{P}^+_{y_1}, \mathcal{P}^+_{y_2}) = 0,$$

for all i. Since the set $\{\mathcal{P}^+_y : y \in Y_s\}$ is a two-dimensional moduli space of stable sheaves on the surface X_s, this is an immediate consequence of the Riemann-Roch theorem [14, Proposition 3.12].

7.3. We now prove Theorem 1.2. Let $\pi\colon X \to S$ be a flat Calabi-Yau fibration of relative dimension two, take a polarisation ℓ on X and let $\hat{\pi}\colon Y \to S$ be a fine, irreducible component of $\mathcal{M}^\ell(X/S)$. Assume Y and X have the same dimension, n say, and let $\mathcal{P}$ be a universal sheaf on $Y \times X$.

The main problem is that we do not know *a priori* that Y is non-singular, so we cannot immediately apply Theorem 3.1. We also do not have any control of the Ext groups between sheaves supported on the singular fibres of X. These problems can be solved by using Theorem 6.1. To apply it, let U be an open subset of S over which π is smooth. Let B be the complement of U in S, a closed subset of S of positive codimension.

Note that $\mathcal{P}_y \otimes \omega_X = \mathcal{P}_y$ for any point $y \in \hat{\pi}^{-1}(U)$ because ω_X restricts to give the trivial bundle on each non-singular fibre of π. By semi-continuity, for any $y \in Y$, there is a non-zero map

$$\mathcal{P}_y \xrightarrow{\theta} \mathcal{P}_y \otimes \omega_X.$$

It is a consequence of Definition 1.1 that the restriction of ω_X to any fibre of π is a numerically trivial line bundle. It follows that the sheaves $\mathcal{P}_y$ and $\mathcal{P}_y \otimes \omega_X$ are stable with the same Hilbert polynomials, so the map θ is an isomorphism.

Let y_1, y_2 be distinct points of Y. The groups $\operatorname{Ext}^i_X(\mathcal{P}_{y_1}, \mathcal{P}_{y_2})$ vanish unless y_1 and y_2 lie on the same fibre of $\hat{\pi}$ and the argument above shows that these groups also vanish when y_1 and y_2 both lie over a point of U. Thus the subscheme $\Gamma(\mathcal{P})$ of the theorem is contained in the union of the diagonal in $Y \times Y$ with the closed subscheme

$$Y \times_S Y \times_S B = \{(y_1, y_2) \in Y \times Y : \hat{\pi}(y_1) = \hat{\pi}(y_2) \in B\}.$$

It only remains to show that this scheme has dimension at most $n+1$. But this is clear since the dimension of B is at most $n-3$ and we assumed that $\hat{\pi}$ is equidimensional so the fibres of $Y \times_S Y \to S$ all have dimension 4.

In the case when π is an elliptic fibration, B has dimension at most $n-2$ and the fibres of $Y \times_S Y \to S$ have dimension 2, so the same argument works.

To show that the fibration $\hat{\pi}\colon Y \to S$ satisfies the condition of Definition 1.1, take a curve $C \subset Y$, contained in a fibre of $\hat{\pi}$, and put $E = \mathcal{O}_C$. Then the object $\Phi(E)$ is supported on a fibre of π. It follows that $\Phi(E) \otimes \omega_X$ has the same numerical invariants as $\Phi(E)$, so using Serre duality

$$\begin{aligned} \chi(E \otimes \omega_Y) &= \chi(E, \mathcal{O}_Y) = \chi(\Phi(E), \Phi(\mathcal{O}_Y)) \\ &= \chi(\Phi(\mathcal{O}_Y), \Phi(E)) = \chi(\mathcal{O}_Y, E) = \chi(E). \end{aligned}$$

This implies that $K_Y \cdot C = 0$. The canonical bundle of the general fibre Y_s of $\hat{\pi}$ is trivial because Φ restricts to give a Fourier-Mukai transform $\Phi_s\colon \mathrm{D}(Y_s) \longrightarrow \mathrm{D}(X_s)$. This completes the proof of Theorem 1.2.

8. FM transforms for elliptic threefolds

In this section we study FM transforms for elliptic threefolds. In many places we follow the treatment of FM transforms for elliptic surfaces given in [4], but in higher dimensions several new ideas are required. In particular we shall prove that Mukai dual fibrations of flat (equidimensional) fibrations are themselves flat.

8.1. Let us fix the following notation.

Definition 8.1. *An elliptic threefold is a Calabi-Yau fibration $\pi\colon X \to S$, as in Definition 1.1, such that S and X have dimensions two and three respectively.*

Let $\pi\colon X \to S$ be an elliptic threefold. Write $f \in H^4(X,\mathbb{Z})$ for the Chern character of the structure sheaf of a non-singular fibre of π and $\tau \in H^6(X,\mathbb{Z})$ for the Chern character of a skyscraper sheaf. Given an object E of $\mathrm{D}(X)$ we put

$$d(E) = \mathrm{c}_1(E) \cdot f$$

and refer to this number as the *fibre degree* of E. Note that $d(E)$ is the degree of the restriction of E to a general fibre of π. Also define $\lambda_{X/S}$ to be the highest common factor of the fibre degrees of objects of $\mathrm{D}(X)$. Equivalently $\lambda_{X/S}$ is the smallest positive integer such that there is a divisor σ on X with $\sigma \cdot f = \lambda_{X/S}$.

8.2. We shall need the following piece of birational geometry.

Proposition 8.2. *Let $\pi_i\colon X_i \to S$ be two elliptic threefolds. Suppose there is a birational equivalence $X_1 \dashrightarrow X_2$ commuting with the maps π_i. Then π_2 is flat if π_1 is.*

Proof. Let $\pi\colon X \to S$ be an elliptic threefold. Let $\Delta \subset S$ be the discriminant locus of π. Thus π is smooth over the open subset $U = S \setminus \Delta$. By Bertini's theorem, we can find a non-singular curve $C \subset S$ meeting each irreducible component of Δ such that $\pi^{-1}(C)$ is non-singular. Since the morphism $\pi\colon \pi^{-1}(C) \to C$ is a relatively minimal elliptic surface, there is a positive integer d such that the natural map

$$\eta\colon \pi^*\pi_*(\omega_X^{\otimes d}) \longrightarrow \omega_X^{\otimes d}$$

is an isomorphism when restricted to $\pi^{-1}(C) \subset X$. This map is also an isomorphism on $\pi^{-1}(U)$ and hence is an isomorphism away from a finite number of fibres of π. It follows that there is a $\mathbb{Q}$-divisor Λ on S such that $K_X = \pi^*\Lambda + D$ where $\pi(D)$ has codimension 2. Thus D is contained in the union of those fibres of π which have dimension 2.

Write $K_{X_i} = \pi_i^*(\Lambda_i) + D_i$. If π_1 is flat then $D_1 = 0$. A simple argument [9, Lemma 1.5] then shows that some positive multiple of D_2 is an effective divisor. If C is a general hyperplane section of this divisor then $K_{X_2} \cdot C < 0$, which is impossible, so $D_2 = 0$. It is then easy to show that π_2 is flat [9, Theorem 2.4]. This completes the proof. $\square$

8.3. The following result is basically a restatement of Theorem 1.2.

Theorem 8.3. *Let $\pi\colon X \to S$ be an elliptic threefold. Take a polarization ℓ on X and let $Y \subset \mathcal{M}^\ell(X/S)$ be a fine, irreducible component of dimension 3 which contains a sheaf supported on a non-singular fibre of π. Let $\mathcal{P}$ be a universal sheaf on $Y \times X$. Then Y with the induced morphism $\hat{\pi}\colon Y \to S$ is an elliptic threefold and the functor $\Phi = \Phi^{\mathcal{P}}_{Y\to X}$ is a FM transform.*

Proof. The only point to note is that we do not require $\hat{\pi}$ to be equidimensional. Since Y is irreducible $\hat{\pi}$ can only have finitely many fibres of dimension 2, so $Y \times_S Y$ has dimension 4 and the argument of Section 7.3 still applies. $\square$

Let $\pi\colon X \to S$ be an elliptic threefold and take a divisor σ on X with $\sigma \cdot f = \lambda_{X/S}$. Fix a pair of integers a and b such that a is positive and b is coprime to $a\lambda_{X/S}$.

If $b = 0$ then $a = \lambda_{X/S} = 1$. If $\Phi\colon \mathrm{D}(Y) \to \mathrm{D}(X)$ is a FM transform taking points of Y to sheaves of Chern character $f - \tau$, then composing with a twist by $\mathcal{O}_X(\sigma)$ gives a FM transform taking points of Y to sheaves of Chern character f.

Thus we may suppose that b is non-zero. Replacing a polarisation ℓ of X by $\sigma \pm bm\ell$ for $m \gg 0$ we may assume that $\ell \cdot f$ is coprime to b. Let Y be the unique irreducible component of $\mathcal{M}^{\ell}(X/S)$ containing a point which represents a stable bundle of rank a and degree b supported on a non-singular fibre of π. As we noted in Section 7.1, the space Y has dimension three and is fine, so Theorem 8.3 shows that there is a FM transform $\Phi\colon \mathrm{D}(Y) \to \mathrm{D}(X)$ taking structure sheaves of points of Y to sheaves of Chern character $af + b\tau$ on X.

8.4. Take notation and assumptions as in Theorem 8.3. Let Ψ denote the functor $\Phi^{\mathcal{Q}}_{X\to Y}$ where

$$\mathcal{Q} = \mathbf{R}\mathcal{H}om_{\mathcal{O}_{Y\times X}}(\mathcal{P}, \pi_X^*\omega_X)[n-1].$$

The inverse of Φ is the functor $\Psi[1]$. The reason for the strange choice of shift in the definition of $\mathcal{Q}$ is the following

Lemma 8.4. *The object $\mathcal{Q}$ is a sheaf on $Y \times X$, flat over Y. If $\hat{\pi}$ is flat then $\mathcal{P}$ and $\mathcal{Q}$ are also flat over X.*

Proof. The sheaf $\mathcal{P}$ is flat over Y, so for every point $(y, x) \in Y \times X$

$$\mathrm{Ext}^i_{Y\times X}(\mathcal{P}, \mathcal{O}_{(y,x)}) = \mathrm{Ext}^i_X(\mathcal{P}_y, \mathcal{O}_x).$$

But $\mathrm{Hom}_X(\mathcal{O}_x, \mathcal{P}_y) = 0$ because $\mathcal{P}_y$ has pure dimension 1, so by Serre duality these groups vanish unless $0 \le i \le 2$. Thus the homological dimension of $\mathcal{P}$ and hence also of $\mathcal{Q}$ is 2. But $\mathcal{Q}$ is supported on $Y \times_S X$ which has codimension 2 in $Y \times X$, so $\mathcal{Q}$ is a sheaf.

Given a point $y \in Y$ let $i_y\colon \{y\} \times X \hookrightarrow Y \times X$ denote the inclusion. The object $\mathcal{Q}_y = \mathbf{L}i_y^*(\mathcal{Q})$ has homological dimension 2 and its support (which is the same as that of $\mathcal{P}_y$) has dimension 1. Thus $\mathcal{Q}_y$ is a sheaf and $\mathcal{Q}$ is flat over Y. If $\hat{\pi}$ is flat, the same argument shows that $\mathcal{P}$ and $\mathcal{Q}$ are flat over X. $\square$

Corollary 8.5. *If $\hat{\pi}$ is flat then for any sheaf E on X and ample line bundle L the sheaf $E \otimes L^n$ is Ψ-WIT_0 for $n \gg 0$.*

Proof. By the last lemma $\mathcal{Q}$ is a sheaf on $Y \times X$, flat over X, so

$$\Psi^i(E \otimes L^n) = \mathbf{R}\pi_{Y,*}(\mathcal{Q} \otimes \pi_X^*(E \otimes L^n)).$$

Since $\pi_X^* L$ is π_Y-ample these groups vanish for $i > 0$ when $n \gg 0$. $\square$

The next lemma is proved by a simple base-change.

Lemma 8.6. *Given a point $s \in S$ such that the fibre X_s is non-singular, let $i_s\colon X_s \hookrightarrow X$ and $j_s\colon Y_s \hookrightarrow Y$ be the corresponding embeddings of schemes. Then there is an isomorphism of functors*

$$\mathbf{L}i_s^* \circ \Phi \cong \Phi_s \circ \mathbf{L}j_s^*$$

where Φ_s is the functor $\Phi^{\mathcal{P}_s}_{Y_s \to X_s}$ and $\mathcal{P}_s$, the restriction of the sheaf $\mathcal{P}$ to $Y_s \times X_s$, is a universal sheaf parameterising stable sheaves on the fibre X_s. $\square$

8.5. We can now prove the main result of this section.

Proposition 8.7. *In the situation of Theorem 8.3, the fibration $\hat{\pi}\colon Y \to S$ is flat precisely if $\pi\colon X \to S$ is.*

Proof. Suppose first that $\hat{\pi}$ is flat. Fix an ample line bundle L on X. Each sheaf $\mathcal{P}_y$ is supported in dimension 1, so for all $y \in Y$ and any integer $i \neq 1$, $\operatorname{Ext}^i_X(L^n, \mathcal{P}_y) = 0$ for $n \gg 0$. Since

$$\operatorname{Hom}^i_{\mathrm{D}(Y)}(\Psi(L^n), \mathcal{O}_y) = \operatorname{Hom}^{i+1}_{\mathrm{D}(X)}(L^n, \mathcal{P}_y)$$

it follows that for $n \gg 0$ the object $\Psi(L^n)$ is a locally-free sheaf on Y.

Given a point $s \in S$ we can use Corollary 8.5 to obtain a Ψ-WIT_0 sheaf F on X whose support is the fibre X_s of π. But

$$\operatorname{Ext}^2_X(L^n, F) = \operatorname{Ext}^2_Y(\Psi(L^n), \Psi(F)) = 0 \text{ for } n \gg 0$$

because $\Psi(F)$ is supported on the fibre Y_s of $\hat{\pi}$ which has dimension 1. This implies that the support of F has dimension 1 and so π is equidimensional, hence flat.

Now suppose that π is flat. For any point $s \in S$ such that the fibre X_s is non-singular the restriction of $\mathcal{P}$ to $Y_s \times X_s$ is a universal sheaf parameterising stable bundles on X_s. It follows from the results of [4, Section 3] that for any $x \in X$ lying on a non-singular fibre of π the sheaf $\mathcal{P}_x$ is a stable bundle on the elliptic curve Y_s of rank a and degree c coprime to $a\lambda_{Y/S}$. As in Section 8.3 we can construct a new elliptic threefold $\pi'\colon Z \to S$ parameterising sheaves of Chern character $af + c\tau$ on Y. The fibrations $\pi\colon X \to S$ and $\pi'\colon Z \to S$ are then birational. Proposition 8.2 shows that π' is flat, so what we proved above shows that $\hat{\pi}\colon Y \to S$ is also flat. This completes the proof. $\square$

9. Further properties of the transforms

In this section we show that the properties of FM transforms for elliptic surfaces listed in [4, Section 6] also hold in the case of flat elliptic threefolds. As an application we prove Theorem 1.4.

9.1. Take notation as in the last section. Putting together what we proved there gives the following generalisation of [4, Theorem 5.3].

Theorem 9.1. *Let $\pi\colon X \to S$ be a flat elliptic threefold. Take an element*

$$\begin{pmatrix} c & a \\ d & b \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}),$$

such that $a > 0$ and $\lambda_{X/S}$ divides d. Then there is a a flat elliptic threefold $\hat{\pi}\colon Y \to S$, and a sheaf $\mathcal{P}$ on $Y \times X$, supported on $Y \times_S X$ and flat over both factors, such that for any point $(y, x) \in Y \times X$, $\mathcal{P}_y$ has Chern character $af + b\tau$ on X and $\mathcal{P}_x$ has Chern character $af + c\tau$ on Y.

The resulting functor $\Phi = \Phi^{\mathcal{P}}_{Y\to X}$ is a FM transform satisfying

$$\begin{pmatrix} r(\Phi E) \\ d(\Phi E) \end{pmatrix} = \begin{pmatrix} c & a \\ d & b \end{pmatrix} \begin{pmatrix} r(E) \\ d(E) \end{pmatrix}, \tag{5}$$

for all objects E of $\mathrm{D}(Y)$.

Proof. The fibration $\hat{\pi}\colon Y \to S$ is constructed as in Section 8.3. Lemma 8.6 and the results of [4, Section 2] show that there exist integers c and d such that (5) holds. It follows that $\lambda_{X/S} = \lambda_{Y/S}$. Twisting $\mathcal{P}$ by pull-backs of line bundles from Y we obtain all possible values of c and d. $\square$

9.2. Take notation and assumptions as in Theorem 9.1. Since $\mathcal{P}$ is flat over Y, with support of dimension one over X, for any sheaf E on Y one has

$$\Phi^i(E) = 0 \text{ unless } 0 \le i \le 1.$$

Furthermore Φ is left-exact so that for any short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

on X one obtains a long exact sequence

$$\begin{aligned} 0 \quad \longrightarrow \quad & \Phi^0(A) \longrightarrow \Phi^0(B) \longrightarrow \Phi^0(C) \\ \longrightarrow \quad & \Phi^1(A) \longrightarrow \Phi^1(B) \longrightarrow \Phi^1(C) \longrightarrow 0. \end{aligned}$$

Similar statements hold for Ψ.

The isomorphism $\Psi \circ \Phi = \mathrm{id}_{\mathrm{D}(Y)}[-1]$ implies that if E is a Φ-WIT$_i$ sheaf on X then $\hat{E}$ is a Ψ-WIT$_{1-i}$ sheaf on Y. More generally, for any sheaf E on Y there is a spectral sequence

$$E_2^{p,q} = \Psi^p(\Phi^q(E)) \implies \begin{cases} E & \text{if } p+q=1 \\ 0 & \text{otherwise.} \end{cases}$$

¿From what we showed above one has $E_2^{p,q} = 0$ unless $p = 0$ or 1 so the spectral sequence degenerates. This leads to the following results.

Lemma 9.2. *Let E be a sheaf on Y. Then $\Phi^0(E)$ is Ψ-WIT$_1$, $\Phi^1(E)$ is Ψ-WIT$_0$, and there is a short exact sequence*

$$0 \longrightarrow \Psi^1(\Phi^0(E)) \longrightarrow E \longrightarrow \Psi^0(\Phi^1(E)) \longrightarrow 0. \quad \square$$

Lemma 9.3. *A sheaf E on Y is Φ-WIT$_0$ if and only if* $\mathrm{Hom}_Y(E, \mathcal{Q}_x) = 0$ *for all $x \in X$.*

Proof. See [4, Lemma 6.5]. $\square$

Lemma 9.4. *If a torsion-free sheaf E on Y is Φ-WIT$_0$, the transformed sheaf $\hat{E}$ is also torsion-free.*

Proof. We repeat the argument of [4, Lemma 7.2]. Note first that the sheaf $\hat{E}$ is Ψ-WIT$_1$. If T is a torsion subsheaf we obtain a short exact sequence

$$0 \longrightarrow T \longrightarrow \hat{E} \longrightarrow Q \longrightarrow 0$$

and applying Ψ gives an exact sequence

$$0 \longrightarrow \Psi^0(Q) \longrightarrow \Psi^1(T) \xrightarrow{g} E \longrightarrow \Psi^1(Q) \longrightarrow 0.$$

Thus T is Ψ-WIT$_1$. Since T is a torsion sheaf, $\mathrm{c}_1(E)$ is the Chern class of an effective divisor, so for any point $y \in Y$

$$\chi(\hat{T}, \mathcal{O}_y) = \chi(T, \mathcal{P}_y) = -\,\mathrm{c}_1(T) \cdot af \le 0.$$

It follows that $\hat{T}$ has rank 0. Since E is torsion-free the map g is zero, so $\Psi^0(Q) = \hat{T}$. But the first sheaf is Φ-WIT$_1$ while the second is Φ-WIT$_0$, so both sheaves are zero and hence $T = 0$. □

Lemma 9.5. *If E is a Φ-WIT$_0$ sheaf on Y whose restriction to the general fibre of $\hat{\pi}$ is stable then the restriction of the transform $\hat{E}$ to the general fibre of π is stable.*

Proof. Lemma 8.6 and [4, Proposition 3.3]. □

9.3. We can now prove Theorem 1.4 which is restated below.

Theorem 1.4. *Let X be a threefold with a flat elliptic fibration $\pi\colon X \to S$ and let $\hat{\pi}\colon Y \to S$ be a Mukai dual fibration as in Theorem 1.2. Let $\mathcal{N}$ be a connected component of the moduli space of rank one, torsion-free sheaves on Y. Then there is a polarisation ℓ of X and a connected component $\mathcal{M}$ of the moduli space of stable torsion-free sheaves on X with respect to ℓ which is isomorphic to $\mathcal{N}$.*

Proof. Given a point $s \in \mathcal{N}$, let $\mathcal{E}_s$ be the corresponding rank 1, torsion-free on Y. The argument of Lemma 8.5 shows that twisting by a line bundle on Y we may assume that all these sheaves are Φ-WIT$_0$. By Lemmas 9.4, 9.5 and 2.1 we can find a polarisation ℓ of X such that each of the transformed sheaves $\hat{\mathcal{E}}_s$ is stable with respect to ℓ. Applying Proposition 3.4 completes the proof. □

In fact we can say slightly more.

Lemma 9.6. *Take assumptions as in Theorem 1.4 and suppose also that $\mathcal{N}$ parameterises ideal sheaves of subschemes $C \subset Y$ which meet each fibre of $\hat{\pi}$ in at most a finite number of points. Then under the isomorphism of the theorem, locally-free sheaves in $\mathcal{M}$ correspond precisely to equidimensional Cohen-Macaulay subschemes $C \subset Y$ of dimension one.*

Proof. For each point $s \in \mathcal{N}$, the corresponding torsion-free sheaf $\mathcal{E}_s$ on Y is of the form $M \otimes \mathcal{I}_C$, where M is a line bundle and $C \subset Y$ is a subscheme of Y. As in the proof of Theorem 1.4 we can assume that M is chosen so that the sheaves $\mathcal{E}_s$ are all Φ-WIT$_0$. There is an isomorphism

$$\mathrm{Ext}^i_X(\hat{\mathcal{E}}_s, \mathcal{O}_x) = \mathrm{Ext}^{i+1}_Y(\mathcal{E}_s, \mathcal{Q}_x)$$

and $\hat{\mathcal{E}}_s$ is locally-free if and only if these groups vanish for all $i > 0$ and all $x \in X$. By Serre duality

$$\mathrm{Ext}^3_Y(\mathcal{E}_s, \mathcal{Q}_x) = \mathrm{Hom}_Y(\mathcal{Q}_x, \mathcal{E}_s \otimes \omega_Y)^\vee = 0$$

because $\mathcal{E}_s$ is torsion-free. Applying the functor $\mathrm{Hom}_Y(-, \mathcal{Q}_x)$ to the exact sequence

$$0 \longrightarrow \mathcal{E}_s \longrightarrow M \longrightarrow M \otimes \mathcal{O}_C \longrightarrow 0$$

and noting that $\mathrm{Ext}^i_Y(M, \mathcal{Q}_x) = 0$ for $i > 1$ shows that

$$\mathrm{Ext}^2_Y(\mathcal{E}_s, \mathcal{Q}_x) = \mathrm{Ext}^3_Y(M \otimes \mathcal{O}_C, \mathcal{Q}_x). = \mathrm{Hom}_Y(\mathcal{Q}_x, M \otimes \mathcal{O}_C)^\vee$$

Since the supports of $\mathcal{Q}_x$ and $\mathcal{O}_C$ intersect in a finite number of points, this group is zero unless there is a non-zero map $\mathcal{O}_y \to \mathcal{O}_C$ for some $y \in Y$. □

Taking $\mathcal{N}$ to be a component of the moduli space of rank one, torsion-free sheaves which parameterise ideal sheaves of zero-dimensional subschemes of Y shows that there are connected components of the moduli space of stable, torsion-free sheaves on X which contain no locally-free sheaves.

10. An example

Let $X \subset \mathbb{P}^2 \times \mathbb{P}^2$ be a non-singular $(3,3)$ (anticanonical) divisor. We shall suppose that X does not contain a fibre of either of the projections $p_i\colon \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^2$. This is equivalent to the assumption that each of the induced projections $\pi_i\colon X \to \mathbb{P}^2$ is flat. Thus X is a Calabi-Yau threefold with two flat elliptic fibrations π_1, π_2. In this section we apply the results of the previous section to compute a moduli space of stable bundles on X.

10.1. By the Lefschetz hyperplane theorem $H^1(X,\mathbb{C}) = 0$ and $H^2(X,\mathbb{C}) = \mathbb{C}^{\oplus 2}$ is generated by the two elements $H_i = \pi_i^*(L)$, where L is the cohomology class of a line on $\mathbb{P}^2$. By Poincaré duality, $H^4(X,\mathbb{C}) = \mathbb{C}^{\oplus 2}$ is generated by the two elements $f_i = \pi_i^*(\omega)$, where ω is the cohomology class of a point on $\mathbb{P}^2$. We have relations

$$H_1^2 = f_1, \quad H_1 \cdot H_2 = f_1 + f_2, \quad H_2^2 = f_2, \quad H_1 \cdot f_2 = H_2 \cdot f_1 = 3.$$

We shall consider X to be elliptically fibred via the map π_1. Thus in the notation of the previous section we take $\pi = \pi_1$ and $f = f_1$. One can check directly that $\pi_*(\mathcal{O}_X)$ and $\mathbf{R}^1\pi_*(\mathcal{O}_X)$ are locally-free and it follows that for any fibre X_s of π one has

$$H^0(X, \mathcal{O}_{X_s}) = H^1(X, \mathcal{O}_{X_s}) = \mathbb{C}.$$

Let $\Delta \subset X \times X$ denote the diagonal and let $\mathcal{P}$ be the kernel of the restriction map $\mathcal{O}_{X\times_S X} \to \mathcal{O}_\Delta$. Thus there is a short exact sequence

$$0 \longrightarrow \mathcal{P} \longrightarrow \mathcal{O}_{X\times_S X} \longrightarrow \mathcal{O}_\Delta \longrightarrow 0. \tag{6}$$

Considered as a sheaf on $X \times X$, $\mathcal{P}$ is flat over both factors, and for any point $x \in X$ there is a short exact sequence

$$0 \longrightarrow \mathcal{P}_x \longrightarrow \mathcal{O}_{X_s} \longrightarrow \mathcal{O}_x \longrightarrow 0, \tag{7}$$

where X_s is the scheme-theoretic fibre of π containing the point x. Put $\Phi = \Phi^{\mathcal{P}}_{X\to X}$.

Lemma 10.1. *For any object E of $\mathrm{D}(X)$ there is a triangle in $\mathrm{D}(X)$*

$$\Phi(E) \longrightarrow \pi^*\mathbf{R}\pi_*(E) \xrightarrow{\eta} E \longrightarrow \Phi(E)[1],$$

where η is the unit of the adjunction $\pi^* \vdash \mathbf{R}\pi_*$.

Proof. The short exact sequence (6) implies that there is a triangle in $\mathrm{D}(X)$

$$\Phi(E) \longrightarrow \Theta(E) \longrightarrow E \longrightarrow \Phi(E)[1].$$

Here Θ is the functor

$$\Theta(-) = \Phi^{\mathcal{O}_{X\times_S X}}_{X\to X} = \mathbf{R}q_{2,*} \circ q_1^*$$

where q_1, q_2 are the projections $X \times_S X \to X$. Base-changing around the diagram

$$\begin{array}{ccc} X \times_S X & \xrightarrow{q_1} & X \\ {\scriptstyle q_2}\Big\downarrow & & {\scriptstyle \pi}\Big\downarrow \\ X & \xrightarrow{\pi} & S \end{array}$$

shows that in fact $\Theta(-) = \pi^* \mathbf{R}\pi_*(-)$. $\square$

As before, let $\Psi[1]$ be a left adjoint for Φ. Then $\Psi \cong \Phi^{\mathcal{Q}}_{X \to X}$ where $\mathcal{Q}$ is a sheaf on $X \times X$, flat over both factors. For each $x \in X$ there is a short exact sequence

$$0 \longrightarrow \mathcal{O}_{X_s} \longrightarrow \mathcal{Q}_x \longrightarrow \mathcal{O}_x \longrightarrow 0.$$

Lemma 10.2. *The functor Φ is a Fourier-Mukai transform.*

Proof. It is not clear that the sheaf $\mathcal{P}_x$ is stable for all $x \in X$ so we shall show that Φ is a FM transform directly. Since $\Psi(\mathcal{O}_x) = \mathcal{Q}_x$ it will be enough to check that for all $x \in X$, $\Phi(\mathcal{Q}_x) = \mathcal{O}_x[-1]$.

By the short exact sequence (7), $H^0(X, \mathcal{P}_x) = 0$ and $H^1(X, \mathcal{P}_x) = \mathbb{C}$. Then Serre duality gives $H^0(X, \mathcal{Q}_x) = \mathbb{C}$ and $H^1(X, \mathcal{Q}_x) = 0$, so $\mathbf{R}\pi_*(\mathcal{Q}_x) = \mathcal{O}_s$ where $s = \pi(x)$. Thus there is a triangle

$$\Phi(\mathcal{Q}_x) \longrightarrow \mathcal{O}_{X_s} \xrightarrow{\eta} \mathcal{Q}_x \longrightarrow \Phi(\mathcal{Q}_x)[1]$$

where the map η is non-zero and hence injective. The result follows. $\square$

10.2. Let $\mathcal{N}$ denote the connected component of the moduli space of rank one, torsion-free sheaves on X containing the ideal sheaf of a fibre of π_2. There is an obvious morphism $\beta\colon \mathbb{P}_2 \to \mathcal{N}$ taking a point in $\mathbb{P}^2$ to the ideal sheaf of the fibre of π_2 lying over it.

Lemma 10.3. *The morphism β is an isomorphism.*

Proof. The morphism β sends a point $t \in T = \mathbb{P}^2$ to the ideal sheaf $E_t = \pi_2^*(\mathcal{I}_t)$. It is clearly injective on points. Using the fact that π_2 is flat it is easy to check that $\operatorname{Ext}^1_X(E_t, E_t)$ is two-dimensional for all $t \in T$, so β is an isomorphism. $\square$

Let M be the line bundle $\mathcal{O}_X(H_2)$. Given an ideal sheaf $\mathcal{I}_C \in \mathcal{N}$ put $E = M \otimes \mathcal{I}_C$. Note that $M \otimes \mathcal{O}_C = \mathcal{O}_C$ so there is a short exact sequence

$$0 \longrightarrow E \longrightarrow M \longrightarrow \mathcal{O}_C \longrightarrow 0. \qquad (8)$$

Lemma 10.4. *The sheaf E is Φ-WIT_0.*

Proof. By Lemma 9.3 we must show that $\operatorname{Hom}_X(E, \mathcal{Q}_x) = 0$ for all $x \in X$. If $\mathcal{Q}_x$ is supported on the fibre D of π, any map $E \to \mathcal{Q}_x$ factors through $E|_D$. There is an exact sequence

$$0 \longrightarrow \operatorname{Tor}_1^{\mathcal{O}_X}(\mathcal{O}_C, \mathcal{O}_D) \longrightarrow E|_D \xrightarrow{g} M \otimes \mathcal{O}_D \longrightarrow \mathcal{O}_C \otimes \mathcal{O}_D \longrightarrow 0.$$

Since $\mathcal{Q}_x$ has no zero-dimensional subsheaves, any map $E|_D \to \mathcal{Q}_x$ factors through the image of g.

Note that $\mathcal{O}_C \otimes \mathcal{O}_D$ is either the structure sheaf of a point or zero, depending on whether C and D meet or not. Thus it will be enough to check that for any $y \in X$ there are no non-zero maps $M \otimes \mathcal{P}_y \to \mathcal{Q}_x$.

Let A be the image of such a map. The fact that A is a subsheaf of $\mathcal{Q}_x$ implies that $H^0(X, A)$ has dimension at most 1 and so $\chi(A) \leq 1$. Similarly $H^1(X, A \otimes M^{-1})$ has dimension at most 1, and since M has fibre degree 3 this gives $\chi(A) \geq 2$, a contradiction. □

10.3. We now compute the Chern character of the transformed sheaf $\hat{E}$. To do this, note first that Lemma 10.1 implies that

$$\operatorname{ch}(\hat{E}) = \pi^* \operatorname{ch}(\mathbf{R}\pi_*(E)) - \operatorname{ch}(E).$$

The sequence (8) together with the fact that $\chi(\mathcal{O}_C) = 0$ give

$$\operatorname{ch}(E) = (1, H_2, \frac{1}{2}f_2, 0) - (0, 0, f_2, 0) = (1, H_2, -\frac{1}{2}f_2, 0).$$

Let $i \colon X \hookrightarrow \mathbb{P}^2 \times \mathbb{P}^2$ be the embedding. Applying the functor $\mathbf{R}p_{1,*}$ to the short exact sequence of sheaves on $\mathbb{P}^2 \times \mathbb{P}^2$

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(-3, -2) \longrightarrow \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(0, 1) \longrightarrow i_* M \longrightarrow 0$$

shows that $\mathbf{R}\pi_{1,*}(M) = \mathcal{O}_S^{\oplus 3}$.

Considered as a subscheme of $\mathbb{P}^2 \times \mathbb{P}^2$, C is of the form $\{x\} \times D$ with $D \subset \mathbb{P}^2$ a curve of degree 3 and $\mathbf{R}p_{1,*}(\mathcal{O}_C) = \mathcal{O}_D$. Thus $\mathbf{R}\pi_{1,*}(E)$ has Chern character

$$(3, 0, 0) - (0, 3, -\frac{9}{2}) = (3, -3, \frac{9}{2})$$

and $\hat{E}$ has Chern character

$$(3, -3H_1, \frac{9}{2}f_1, 0) - (1, H_2, -\frac{1}{2}f_2, 0) = (2, -3H_1 - H_2, \frac{9}{2}f_1 + \frac{1}{2}f_2, 0).$$

To tidy this up, let L be the line bundle $\mathcal{O}_X(2H_1 + H_2)$. Then

$$\operatorname{ch}(L) = (1, 2H_1 + H_2, 4f_1 + \frac{5}{2}f_2, 9)$$

so that $\hat{E} \otimes L$ has Chern character $(2, H_1 + H_2, \frac{3}{2}f_1 - \frac{1}{2}f_2, 0)$.

By Lemmas 9.5 and 9.6 the sheaf $\hat{E}$ is locally-free with stable restriction to the general fibre of π. Applying Lemmas 2.1, 2.2 and Proposition 3.4 gives

Proposition 10.5. *There is a polarisation ℓ on X and a connected component of the moduli space of stable sheaves with respect to ℓ of Chern character*

$$(2, H_1 + H_2, \frac{3}{2}f_1 - \frac{1}{2}f_2, 0)$$

which is isomorphic to $\mathbb{P}^2$. Moreover, all elements of this component are μ-stable vector bundles. □

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