STABILITY CONDITIONS AND THE A_2 QUIVER

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ABSTRACT. We compute the space of stability conditions on the CY_n version of the A_2 quiver for all n and relate it to the Frobenius-Saito structure on the unfolding space of the A_2 singularity.

1. Introduction

In this paper we study spaces of stability conditions $\mathrm{Stab}(\mathrm{D}_n)$ on the sequence of CY_n triangulated categories D_n associated to the A_2 quiver. Our main result is Theorem 1.1 below. There are several striking features. Firstly we obtain uniform results for all n: the space of stability conditions quotiented by the action of the spherical twists is independent of n, although the identification maps are highly non-trivial. Secondly, there is a close link between our spaces of stability conditions and the Frobenius-Saito structure on the unfolding space of the A_2 singularity: in fact this structure is precisely what encodes the identifications between our stability spaces for various n. A third interesting feature is that the space of stability conditions on the usual derived category of the A_2 quiver arises as a kind of limit of the spaces for the categories D_n as $n \to \infty$.

1.1. **Statement of results.** For any integer $n \geqslant 2$ we let $\mathrm{D}_n = \mathrm{D}_{\mathrm{CY}_n}(A_2)$ denote the bounded derived category of the CY_n complex Ginzburg algebra associated to the A_2 quiver. It is a triangulated category of finite type over $\mathbb{C}$ and is characterised by the following two properties:

(a) It is CY_n, i.e. for any pair of objects $A, B \in \mathrm{D}_n$ there are natural isomorphisms

$$\mathrm{Hom}^*_{\mathrm{D}_n}(A, B) \cong \mathrm{Hom}^*_{\mathrm{D}_n}(B, A[n])^\vee.$$

(b) It is (classically) generated by two spherical objects S_1, S_2 satisfying

$$\mathrm{Hom}^*_{\mathrm{D}_n}(S_1, S_2) = \mathbb{C}[-1]. \quad (1)$$

We denote by D_∞ the usual bounded derived category of the A_2 quiver. It is again a $\mathbb{C}$-linear triangulated category, and is characterised by the property that it is generated by two exceptional objects S_1, S_2 satisfying (1) and

$$\mathrm{Hom}^*_{\mathrm{D}_\infty}(S_2, S_1) = 0.$$

The notation D_∞ is convenient: the point is that as n increases the Serre dual to the extension $S_1 \to S_2[1]$ occurs in higher and higher degrees until when $n = \infty$ it doesn't occur at all.

For $2 \leqslant n \leqslant \infty$ we let $\operatorname{Stab}(\mathrm{D}_n)$ denote the space of stability conditions on the category D_n. We let $\operatorname{Stab}_*(\mathrm{D}_n) \subset \operatorname{Stab}(\mathrm{D}_n)$ be the connected component containing stability conditions in which the objects S_1 and S_2 are stable of equal phase. Let $\operatorname{Aut}(\mathrm{D}_n)$ denote the group of exact $\mathbb{C}$-linear autoequivalences of the category D_n, considered up to isomorphism of functors. We let $\operatorname{Aut}_*(\mathrm{D}_n)$ denote the subquotient consisting of autoequivalences which preserve the connected component $\operatorname{Stab}_*(\mathrm{D}_n)$ modulo those which act trivially on it. When $n < \infty$ we let $\operatorname{Sph}_*(\mathrm{D}_n)$ denote the subgroup of $\operatorname{Aut}_*(\mathrm{D}_n)$ generated by the Seidel-Thomas twist functors Tw_{S_1} and Tw_{S_2} corresponding to the spherical objects S_1 and S_2.

The Cartan algebra of the Lie algebra $\mathfrak{sl}_3$ corresponding to the A_2 root system can be described explicitly as

$$\mathfrak{h} = \{(u_1, u_2, u_3) \in \mathbb{C}^3 : \sum_i u_i = 0\}.$$

The complement of the root hyperplanes is

$$\mathfrak{h}^{\text{reg}} = \{(u_1, u_2, u_3) \in \mathfrak{h} : i \neq j \implies u_i \neq u_j\}.$$

There is an obvious action of the Weyl group $W = S_3$ permuting the u_i which is free on $\mathfrak{h}^{\text{reg}}$. The quotient $\mathfrak{h}/W$ is isomorphic to $\mathbb{C}^2$ with co-ordinates (a, b) by setting

$$p(x) = (x - u_1)(x - u_2)(x - u_3) = x^3 + ax + b.$$

The image of the root hyperplanes $u_i = u_j$ is the discriminant

$$\Delta = \{(a, b) \in \mathbb{C}^2 : 4a^3 + 27b^2 = 0\}.$$

We can now state the main result of this paper.

Theorem 1.1. *(a) For $2 \leqslant n < \infty$ there is an isomorphism of complex manifolds*

$$\operatorname{Stab}_*(\mathrm{D}_n)/\operatorname{Sph}_*(\mathrm{D}_n) \cong \mathfrak{h}^{\text{reg}}/W.$$

Under this isomorphism the central charge map $\operatorname{Stab}_*(\mathrm{D}_n) \to \mathbb{C}^2$ induces the multi-valued map $\mathfrak{h}^{\text{reg}}/W \to \mathbb{C}^2$ given by

$$\int_{\gamma_i} p(x)^{(n-2)/2}\, dx$$

for an appropriate basis of paths γ_i connecting the zeroes of the polynomial $p(x)$.

(b) For $n = \infty$ there is an isomorphism of complex manifolds

$$\mathrm{Stab}(\mathrm{D}_\infty) \cong \mathfrak{h}/W.$$

Under this isomorphism the central charge map $\mathrm{Stab}(\mathrm{D}_\infty) \to \mathbb{C}^2$ *corresponds to the map* $\mathfrak{h}/W \to \mathbb{C}^2$ *given by*

$$\int_{\delta_i} e^{p(x)}\, dx$$

for an appropriate basis of paths δ_i which approach ∞ in both directions along rays for which $x^3 \to -\infty$.

Theorem 1.1 gives a precise link with the Frobenius-Saito structure on the unfolding space of the A_2 singularity $x^3 = 0$. The corresponding Frobenius manifold is precisely $M = \mathfrak{h}/W$. The maps appearing in part (a) of our result are then the *twisted period maps* of M with parameter $\nu = (n-2)/2$ (see Equation (5.11) of [4]). The map in part (b) is given by the *deformed flat co-ordinates* of M with parameter $\hbar = 1$ (see [3, Theorem 2.3]).

1.2. **Relation with previous work.** Just as we were failing to get round to finishing this paper, A. Ikeda posted [6] on the arxiv which also proves Theorem 1.1 (a), and indeed generalizes it to the case of the A_k quiver for all $k \geqslant 1$. The methods we use here are quite different however, and also yield (b), so we feel that this paper is also worth publishing.

As explained above, two of the most interesting features of Theorem 1.1 are the fact that the space $\mathrm{Stab}\,{*}(\mathrm{D}_n)/\,\mathrm{Sph}_*(\mathrm{D}_n)$ is independent of n, and that it embeds in $\mathrm{Stab}_*(\mathrm{D}_n)$. At the level of exchange graphs such results were observed for arbitrary acyclic quivers by one of us with A.D. King [8].

The $n < \infty$ case of Theorem 1.1 was first considered by R.P. Thomas in [15]: he obtained the $n = 2$ case and discussed the relationship with Fukaya categories and homological mirror symmetry. The $n = 2$ case was also proved in [1] and generalised to arbitrary ADE Dynkin diagrams. The $n = 3$ case of Theorem 1.1 was proved in [13], and was extended to all Dynkin quivers of A and D type in [2]. The first statement of part (a), that $\mathrm{Stab}(\mathrm{D}) \cong \mathfrak{h}^{\mathrm{reg}}/W$, was proved for all $n < \infty$ in [10].

The case $n = \infty$ of Theorem 1.1 was first considered by King [7] who proved that $\mathrm{Stab}(\mathrm{D}_\infty) \cong \mathbb{C}^2$. This result was obtained by several other researchers since then, and a proof was written down in [10]. The more precise statement of Theorem 1.1 (b) was conjectured by A. Takahashi [14].

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2. Auto-equivalences and t-structures

In this section we describe the principal components of the exchange graphs of the categories $\mathrm{D}_n = \mathrm{D}_{\mathrm{CY}_n}(A_2)$ and study the action of the group of reachable auto-equivalences. We start by recalling some general definitions concerning tilting.

2.1. Let D be a triangulated category. We shall be concerned with bounded t-structures on D. Any such t-structure is determined by its heart $\mathcal{A} \subset \mathrm{D}$, which is a full abelian subcategory. We use the term *heart* to mean the heart of a bounded t-structure. A heart will be called *finite-length* if it is artinian and noetherian as an abelian category.

We say that a pair of hearts $(\mathcal{A}_1, \mathcal{A}_2)$ in D is a *tilting pair* if the equivalent conditions

$$\mathcal{A}_2 \subset \langle \mathcal{A}_1, \mathcal{A}_1[-1] \rangle, \quad \mathcal{A}_1 \subset \langle \mathcal{A}_2[1], \mathcal{A}_2 \rangle$$

are satisfied. Here the angular brackets signify the extension-closure operation We also say that $\mathcal{A}_1$ is a *left tilt* of $\mathcal{A}_2$, and that $\mathcal{A}_2$ is a *right tilt* of $\mathcal{A}_1$. Note that $(\mathcal{A}_1, \mathcal{A}_2)$ is a tilting pair precisely if so is $(\mathcal{A}_2[1], \mathcal{A}_1)$.

If $(\mathcal{A}_1, \mathcal{A}_2)$ is a tilting pair in D, then the subcategories

$$\mathcal{T} = \mathcal{A}_1 \cap \mathcal{A}_2[1], \quad \mathcal{F} = \mathcal{A}_1 \cap \mathcal{A}_2$$

form a torsion pair $(\mathcal{T}, \mathcal{F}) \subset \mathcal{A}_1$. Conversely, if $(\mathcal{T}, \mathcal{F}) \subset \mathcal{A}_1$ is a torsion pair, then the subcategory $\mathcal{A}_2 = \langle \mathcal{F}, \mathcal{T}[-1] \rangle$ is a heart, and the pair $(\mathcal{A}_1, \mathcal{A}_2)$ is a tilting pair.

A special case of the tilting construction will be particularly important. Suppose that $\mathcal{A}$ is a finite-length heart and $S \in \mathcal{A}$ is a simple object. Let $\langle S \rangle \subset \mathcal{A}$ be the full subcategory consisting of objects $E \in \mathcal{A}$ all of whose simple factors are isomorphic to S. Define full subcategories

$$S^{\perp} = \{E \in \mathcal{A} : \mathrm{Hom}_{\mathcal{A}}(S, E) = 0\}, \quad {}^{\perp}S = \{E \in \mathcal{A} : \mathrm{Hom}_{\mathcal{A}}(E, S) = 0\}.$$

One can either view $\langle S \rangle$ as the torsion part of a torsion pair on $\mathcal{A}$, in which case the torsion-free part is $S^{\perp}$, or as the torsion-free part, in which case the torsion part is ${}^{\perp}S$. We can then define tilted hearts

$$\mu_S^-(\mathcal{A}) = \langle S[1], {}^{\perp}S \rangle, \qquad \mu_S^+(\mathcal{A}) = \langle S^{\perp}, S[-1] \rangle,$$

which we refer to as the left and right tilts of the heart $\mathcal{A}$ at the simple S. They fit into tilting pairs $(\mu_S^-(\mathcal{A}), \mathcal{A})$ and $(\mathcal{A}, \mu_S^+(\mathcal{A}))$. Note the relation

$$\mu_{S[1]}^+ \circ \mu_S^-(\mathcal{A}) = \mathcal{A}.$$

The *exchange graph* $\mathrm{EG}(\mathrm{D})$ is the graph with vertices the finite-length hearts in D and edges corresponding to simple tilts. The group $\mathrm{Aut}(\mathrm{D})$ of triangulated auto-equivalences of D acts on this graph in the obvious way.

2.2. For any $2 \leqslant n \leqslant \infty$ the category $\mathrm{D}_n = \mathrm{D}_{\mathrm{CY}_n}(A_2)$ has a canonical finite-length heart $\mathcal{A}_n \subset \mathrm{D}_n$ which is the extension-closed subcategory generated by S_1 and S_2. We denote by $\mathrm{EG}^o(\mathrm{D}_n)$ the *principal component* of the exchange graph, i.e. the connected component containing the canonical heart $\mathcal{A}_n \subset \mathrm{D}_n$. The vertices of $\mathrm{EG}^o(\mathrm{D}_n)$ are called *reachable* hearts. We say that a heart $\mathcal{A} \subset \mathrm{D}_n$ is *full* if it is equivalent to the canonical heart $\mathcal{A}_n$ as an abelian category.

Remark 2.1. When $n > 2$, this canonical heart $\mathcal{A}_n$ is equivalent to the category $\mathrm{Rep}(A_2)$ of representations of the A_2 quiver; besides the simple objects S_1 and S_2, it contains one more indecomposable object which we denote E; there is a short exact sequence

$$0 \longrightarrow S_2 \longrightarrow E \longrightarrow S_1 \longrightarrow 0. \tag{2}$$

When $n = 2$, the canonical heart $\mathcal{A}_2$ is equivalent to the category of representations of the preprojective algebra of the A_2 quiver: besides E there is another non-simple indecomposable fitting into a short exact sequence

$$0 \longrightarrow S_1 \longrightarrow F \longrightarrow S_2 \longrightarrow 0. \tag{3}$$

A triangulated autoequivalence of D_n is called *reachable* if its action on $\mathrm{EG}(\mathrm{D}_n)$ preserves the connected component $\mathrm{EG}^o(\mathrm{D}_n)$. We write $\mathrm{Aut}_*(\mathrm{D}_n)$ for the sub-quotient of the group $\mathrm{Aut}(\mathrm{D})$ consisting of reachable auto-equivalences modulo those which act trivially on $\mathrm{EG}^o(\mathrm{D}_n)$. We will show that this agrees with the definition given in the introduction later (see Remark 4.3(b)).

Lemma 2.2. *Let $2 \leqslant n < \infty$ and define the following auto-equivalences of D_n:*

$$\Sigma = (\mathrm{Tw}_{S_1} \mathrm{Tw}_{S_2})[n-1], \quad \Upsilon = (\mathrm{Tw}_{S_2} \mathrm{Tw}_{S_1} \mathrm{Tw}_{S_2})[2n-3].$$

Then we have

$$\Sigma(S_1, E, S_2) = (S_2[1], S_1, E), \quad \Upsilon(S_1, S_2) = (S_2, S_1[n-2]).$$

Proof. For any spherical object S we always have $\mathrm{Tw}_S(S) = S[1-n]$, and for any pair of spherical objects we have the relation

$$\mathrm{Tw}_{S_1} \circ \mathrm{Tw}_{S_2} = \mathrm{Tw}_{\mathrm{Tw}_{S_1}(S_2)} \circ \mathrm{Tw}_{S_1}.$$

The defining property (1) together with the short exact sequence (2) shows that

$$\mathrm{Tw}_{S_1}(S_2) = E, \quad \mathrm{Tw}_E(S_1) = S_2[1], \quad \mathrm{Tw}_{S_2}(E) = S_1.$$

Thus $\Sigma = (\mathrm{Tw}_E \circ \mathrm{Tw}_{S_1})[n-1]$. Hence

$$\Sigma(S_1) = \mathrm{Tw}_E(S_1) = S_2[1], \quad \Sigma(S_2) = \mathrm{Tw}_{S_1}(S_2) = E.$$

It follows that $\Sigma(E)$ is the unique non-trivial extension of these two objects, namely S_1.

Moving on to the second identity we know that Tw_{S_1} and Tw_{S_2} satisfy the braid relation (see Prop. 2.7 below). Hence

$$\Upsilon(S_1) = \Sigma(S_1[-1]) = S_2, \quad \Upsilon(S_2) = \mathrm{Tw}_{S_2}(E[n-2]) = S_1[n-2].$$

This completes the proof. □

2.3. The following description of the tilting operation in D_n is the combinatorial underpinning of our main result.

Proposition 2.3. *Let $2 \leqslant n \leqslant \infty$, and consider hearts obtained by performing simple tilts of the standard heart $\mathcal{A} \subset \mathrm{D}_n$.*

(a) The left tilt of $\mathcal{A}$ at the simple S_2 is another full heart:

$$\mathcal{A} = \langle S_1, S_2 \rangle \to \langle S_2[1], E \rangle = \Sigma(\mathcal{A}).$$

(b) If $2 < n < \infty$ then repeated left tilts at appropriate shifts of S_1 gives a sequence of hearts

$$\mathcal{A} = \langle S_1, S_2 \rangle \to \langle S_1[1], S_2 \rangle \to \langle S_1[2], S_2 \rangle \to \cdots \to \langle S_1[n-2], S_2 \rangle = \Upsilon(\mathcal{A}).$$

Proof. This is easily checked by hand, or one can consult [8, Proposition 5.4]. □

Remarks 2.4. (a) The statement of part (b) deserves individual comment in each of the cases $n = 2, 3, \infty$:

(i) if $n = \infty$ the sequence of non-full hearts is of course infinite;

(ii) if $n = 3$ the first tilt is already a full heart so no non-full hearts arise;

(iii) if $n = 2$, the statement of Prop. 2.3 (b) needs slight modification: there is now a non-trivial extension (3) and the left tilt of $\mathcal{A}$ at S_1 is

$$\mathcal{A} = \langle S_1, S_2 \rangle \to \langle F, S_1[-1] \rangle = \Sigma^*(\mathcal{A}),$$

where $\Sigma^* = (\mathrm{Tw}_{S_2} \mathrm{Tw}_{S_1})[1]$. Once again no non-full hearts arise.

(b) When $n > 3$ the intermediate hearts in the sequence in (b) are non-full. In fact, since

$$\mathrm{Hom}^1_{\mathrm{D}_n}(S_1[k], S_2) = 0 = \mathrm{Hom}^1_{\mathrm{D}_n}(S_2, S_1[k])$$

for $0 < k < n-2$, each of these hearts is equivalent to the category of representations of the quiver with two vertices and no arrows.

Corollary 2.5. *The auto-equivalences* Σ, Υ *and* $[1]$ *are all reachable. In the group* $\operatorname{Aut}_*(\mathrm{D}_n)$ *we have relations*

$$\Sigma^3 = [1], \quad \Upsilon^2 = [n-2],$$

Proof. The reachability of Σ and Υ is immediate from the last result. Since the twist functors Tw_{S_i} are reachable it follows that $[n-1]$ and $[2n-3]$ and hence also $[1]$ are reachable. From Lemma 2.2 we know that the auto-equivalence $\Sigma^3[-1]$ fixes the objects S_1, S_2. This is enough to ensure that it acts trivially on $\mathrm{EG}^0(\mathrm{D}_n)$ and hence defines the identity element in $\operatorname{Aut}_*(\mathrm{D}_n)$. Similarly for $\Upsilon^2[2-n]$. □

Proposition 2.6. *For* $2 \leqslant n \leqslant \infty$ *the action of* $\operatorname{Aut}_*(\mathrm{D}_n)$ *on the set of full reachable hearts is free and transitive.*

Proof. It is free by definition. That it is transitive follows from the characterisation of the categories D_n in terms of generators, or by the explicit description of tilts given in Prop. 2.3. □

2.4. We denote by Br_3 the Artin braid group of the A_2 root system; it is the fundamental group of $\mathfrak{h}^{\mathrm{reg}}/W$. More concretely, Br_3 is the standard braid group on 3 strings and has a presentation

$$\mathrm{Br}_3 = \langle \sigma_1, \sigma_2 : \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle.$$

The centre of Br_3 is generated by the element $\tau = (\sigma_1\sigma_2)^3$ and there is a short exact sequence

$$1 \longrightarrow \mathbb{Z} \xrightarrow{\tau} \mathrm{Br}_3 \longrightarrow \mathrm{PSL}(2,\mathbb{Z}) \longrightarrow 1.$$

We can give the following description of the group $\operatorname{Aut}_*(\mathrm{D}_n)$.

Proposition 2.7. *Let* $2 \leqslant n < \infty$.

- *(a) The group* $\operatorname{Aut}_*(\mathrm{D}_n)$ *is generated by the subgroup* $\operatorname{Sph}_*(\mathrm{D}_n)$ *together with the shift functor* $[1]$.
- *(b) There is an isomorphism* $\mathrm{Br}_3 \cong \operatorname{Sph}_*(\mathrm{D}_n)$ *sending the generator* σ_i *to* Tw_{S_i}.
- *(c) The isomorphism in (b) sends the central element* τ *to* $[4-3n]$.
- *(d) The smallest power of* $[1]$ *contained in* $\operatorname{Sph}_*(\mathrm{D}_n)$ *is* $[4-3n]$. *Thus there is a short exact sequence*

$$1 \longrightarrow \operatorname{Sph}_*(\mathrm{D}_n) \longrightarrow \operatorname{Aut}_*(\mathrm{D}_n) \longrightarrow \mu_{3n-4} \longrightarrow 1.$$

Proof. Part (a) follows from the explicit description of tilts given in Prop. 2.3 since any element of $\operatorname{Aut}_*(\mathrm{D}_n)$ takes the canonical heart $\mathcal{A}$ to a reachable full heart. Part (b) was proved by Seidel and Thomas [12]. Part (c) is immediate from Cor. 2.5. Part (d) then follows from the fact that τ generates the centre of Br_3, since any shift $[d]$ lying in $\operatorname{Sph}_*(\mathrm{D})$ is necessarily central and hence corresponds to a multiple of τ. □

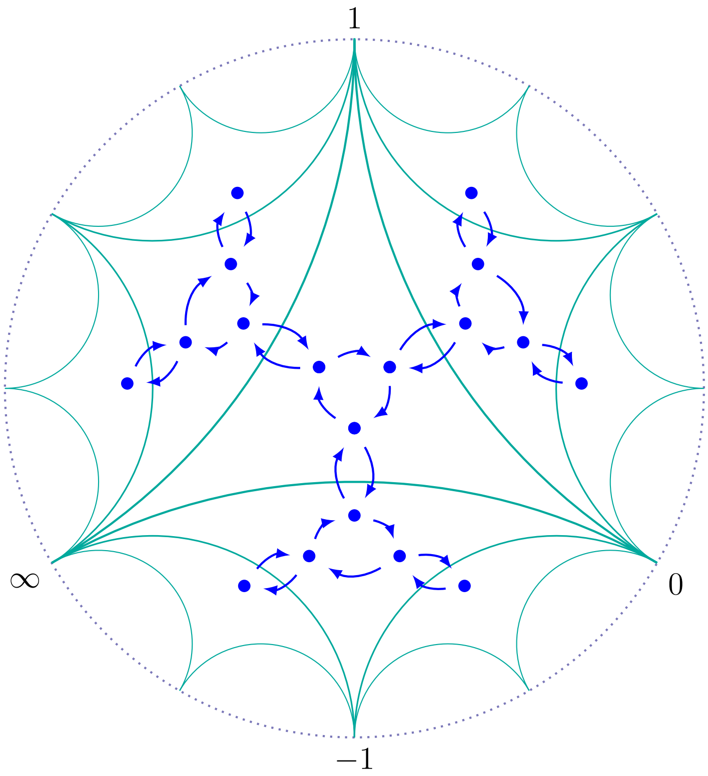

FIGURE 1. The projective exchange graph of D_3 drawn on the hyperbolic disc. The action of $\mathbb{P}\,\mathrm{Aut}_*(\mathrm{D}_3)$ corresponds to the standard action of $\mathrm{PSL}(2,\mathbb{Z})$ on the disc.

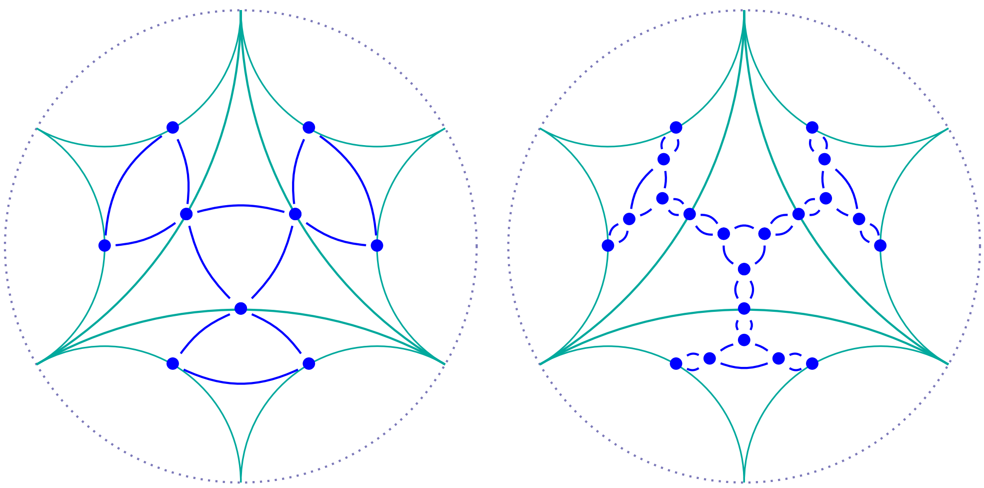

FIGURE 2. Similar pictures of the projective exchange graphs of D_2 and D_4 (orientations omitted). As before, the action of $\mathbb{P}\,\mathrm{Aut}_*(\mathrm{D}_n) \cong \mathrm{PSL}(2,\mathbb{Z})$ corresponds to the standard one.

It will be useful to introduce the quotient group

$$\mathbb{P}\operatorname{Aut}_*(\mathrm{D}_n) = \operatorname{Aut}_*(\mathrm{D}_n)/[1].$$

When $2 \leqslant n < \infty$ it follows from Prop. 2.7 that

$$\mathbb{P}\operatorname{Aut}_*(\mathrm{D}_n) = \operatorname{Sph}_*(\mathrm{D}_n)/\langle[4-3n]\rangle \cong \operatorname{Br}_3/\langle\tau\rangle \cong \operatorname{PSL}(2,\mathbb{Z}).$$

Note that by Cor. 2.5, the autoequivalences Σ and Υ define elements of $\mathbb{P}\operatorname{Aut}_*(\mathrm{D}_n)$ of orders 2 and 3 respectively.

2.5. The auto-equivalence group of the category D_∞ is much simpler.

Proposition 2.8. *There is an equality* $\operatorname{Aut}_*(\mathrm{D}_\infty) = \operatorname{Aut}(\mathrm{D}_\infty)$. *Moreover*

- *(a) The group* $\operatorname{Aut}(\mathrm{D}_\infty) \cong \mathbb{Z}$ *with the Serre functor* Σ *being a generator.*
- *(b) There is a relation* $\Sigma^3 = [1]$.

Proof. This is easy and well-known. The Auslander-Reiten quiver for D_∞ is an infinite strip

$$\cdots \to E[-1] \to S_1[-1] \to S_2 \to E \to S_1 \to S_2[1] \to E[1] \to \cdots$$

and Σ moves along this to the right by one place. □

It follows that $\mathbb{P}\operatorname{Aut}(\mathrm{D}_\infty) \cong \mu_3$. Note that our use of the symbol Σ in Prop. 2.8 is reasonably consistent with our earlier use for the category D_n: for example the first part of Cor. 2.5 continues to hold in the $n = \infty$ case.

3. Conformal maps

In this section we describe some explicit conformal maps which will be the analytic ingredients in the proof of our main result.

3.1. For $2 \leqslant n < \infty$ we consider the domain $R_n \subset \mathbb{C} \subset \mathbb{P}^1$ depicted in Figure 3. It is bounded by the line $\operatorname{Re}(z) = (2-n)/2$ and by the curves $\ell_\pm$ which are the images under the map $z \mapsto (1/\pi i)\log(z)$ of the arcs of circles of Apollonius

$$r_\pm = \{z \in \mathbb{C} : |z^{\pm 1} + 1| = 1\}. \tag{4}$$

connecting 0 and $\omega^{\pm 1} = e^{\pm 2\pi i/3}$. We also consider splitting R_n into two halves $R_n^\pm$ by dividing it along the line $\operatorname{Im}(z) = 0$, and we take R_n^+ to be the part lying below the real axis. Note that the boundary of the domain R_n^+ has three singular points: $(2-n)/2, 2/3$ and ∞. The Riemann mapping theorem ensures that there is a unique biholomorphism

$$f_n \colon \mathcal{H} \to R_n^+$$

which extends continuously over the boundary of $\mathcal{H} \subset \mathbb{P}^1$ and sends $(0,1,\infty)$ to $(\frac{2-n}{2}, \infty, \frac{2}{3})$.

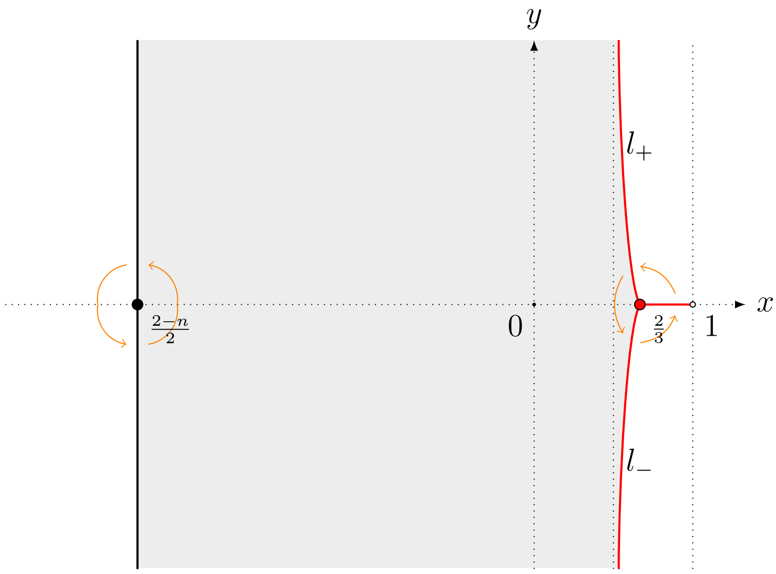

FIGURE 3. The region R_n

Proposition 3.1. *For $n < \infty$ the functions f_n can be explicitly written as*

$$f_n(t) = \frac{1}{\pi i} \log\left(\frac{\phi_n^{(2)}(a,b)}{\phi_n^{(1)}(a,b)}\right)$$

where $t = -(27b^2)/(4a^3)$ and

$$\phi_n^{(i)}(a,b) = \int_{\gamma_i} (x^3 + ax + b)^{\frac{n-2}{2}} dx,$$

for appropriately chosen paths γ_i connecting zeroes of the integrand.

Note that the function f_n only depends on $t = -(27b^2)/(4a^3)$ because rescaling (a,b) with weights $(4,6)$ rescales both functions $\phi_n^{(i)}$ with weight $3(n-2)+2 = 3n-4$ and leaves their ratio unchanged.

Proof. The function $g_n(t) = \exp \pi i f_n(t)$ is holomorphic on the upper half-plane $\mathcal{H} \subset \mathbb{P}^1$ and extends continuously over its boundary. The images in $\mathbb{P}^1$ of the three sides of R_n^+ under the map $z \mapsto \exp(\pi i z)$ are arcs of circles in $\mathbb{P}^1$ connecting the points $((-1)^n, 0, \omega)$. Thus g_n maps the three components of $\mathbb{R} \setminus \{0,1\}$ to these three arcs. It is not the case that the upper half-plane $\mathcal{H}$ is mapped into the interior of the curvilinear triangle bounded by these arcs, but nonetheless, the usual proof of the Schwarz triangle theorem, as in e.g. [9, §V.7], applies almost verbatim and shows that g_n is given by the ratio of a pair of linearly independent solutions to a second-order linear differential equation with precisely three regular singularities at $(0, 1, \infty)$. The differences between the characteristic exponents of the equation at these singular points are then determined by the angles subtended by the circular arcs at the corresponding points $((-1)^n, \infty, \omega)$ where they meet. The only point which requires any extra argument compared to the standard proof of the Schwarz triangle theorem is the local behaviour of the function $g_n(t)$ at $t = 1$; this is dealt with by Lemma 3.2 below.

We conclude that g_n is given by the ratio of a pair of linearly independent solutions to any second-order linear differential equation with precisely three regular singularities at $(0, 1, \infty)$ and differences of characteristic exponents $(\frac{1}{2}, \frac{(n-1)}{2}, \frac{1}{3})$. Consider now pulling back the differential equation satisfied by $g_n(t)$ along the double cover $t = (2z - 1)^2$ branched at $(0, \infty)$. This has the effect of removing the singularity at $t = 0$, and the resulting differential equation has three regular singularities at $(0, 1, \infty)$ with differences in characteristic exponents $(\frac{n-1}{2}, \frac{n-1}{2}, \frac{2}{3})$. We prove in the Appendix that the periods $\phi_n(z) = \phi_n(-3, 2(2z - 1))$ satisfy precisely such a differential equation. This completes the proof. □

Lemma 3.2. *In a neighbourhood of $t = 1$ there is an expression*

$$T \circ g_n(t) = (t - 1)^{(n-1)/2} \cdot k(t)$$

where T is a Möbius transformation fixing $g_n(1) = 0$ and $k(t)$ is holomorphic at $t = 1$ and real-valued for t real.

Proof. The intervals $(0, 1]$ and $[1, \infty)$ are mapped by g_n to two arcs of circles which meet at 0 at an angle of $(n - 1)\pi/2$. We take a Möbius transformation T which fixes $g_n(1) = 0$ and maps these two arcs of circles to rays $\mathbb{R}_{\geqslant 0} \cdot i^{n-1}$ and $\mathbb{R}_{\geqslant 0}$ respectively. Consider now the function

$$h_n(t) = T(g_n(t))^{2/(n-1)},$$

defined by some branch of log near 0. Then h_n is holomorphic in the upper-half plane and extends continuously to a real-valued function on the intervals $(0, 1)$ and $(1, \infty)$. The last thing to check is that it is continuous at $t = 1$: given this, the reflection principle shows that h_n is holomorphic at $t = 1$ and the usual argument can be applied. For the claim note that if $H(z) = z^{(n-1)/2}$ then $H^{-1} \circ T \circ H$ is holomorphic at $z = 0$. Thus it is enough to consider $H^{-1} \circ g_n(t) = \exp(2\pi i f_n(t)/(n - 1))$. But this is continuous at $t = 1$ because f_n is. □

3.2. In the case $n = \infty$ we consider the region R_∞ depicted in Figure 4. It is bounded by the same two curves $\ell_\pm$. We again consider the half region R_∞^+ consisting of points of

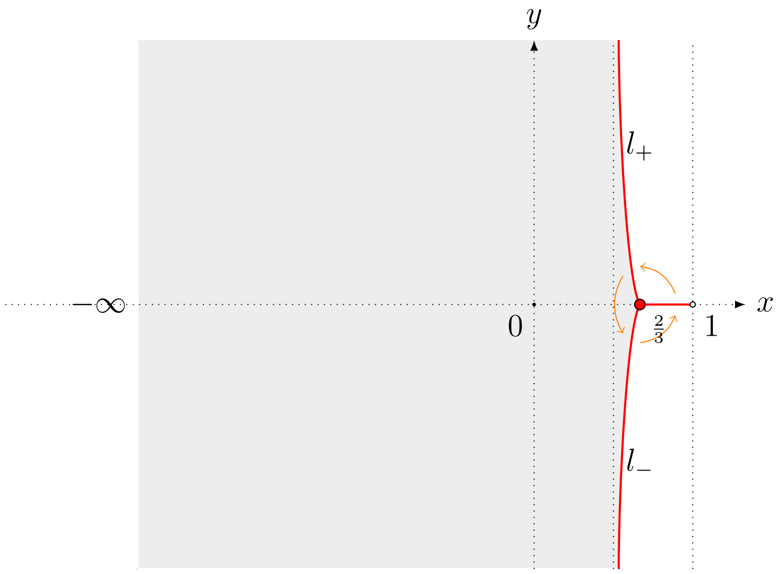

FIGURE 4. The region R_∞

R_∞ with positive imaginary part. This region R_∞^+ has just two vertices: $2/3$ and ∞. The Riemann mapping theorem ensures that there is a biholomorphism

$$f_\infty \colon \mathcal{H} \to R_\infty^+$$

which extends continuously over the boundary, and sends the points $(0, \infty)$ to $(\frac{2}{3}, \infty)$. This map is unique up to precomposing by a map of the form $t \mapsto \lambda \cdot t$ with λ real. Considering the orientations of the two regions shows that $\mathbb{R}_{>0}$ is mapped to ℓ_+ and $\mathbb{R}_{<0}$ to the open interval of the real axis $(-\infty, 2/3)$.

Proposition 3.3. *The function f_∞ can be written explicitly as*

$$f_\infty(t) = \frac{1}{\pi i} \log \left(\frac{\phi_\infty^{(2)}(a,b)}{\phi_\infty^{(1)}(a,b)} \right)$$

where $t = a^3$, b is arbitrary, and

$$\phi_\infty^{(i)}(a,b) = \int_{\delta_i} e^{x^3 + ax + b}\, dx,$$

for an appropriate basis of paths δ_i which approach ∞ in both directions along rays for which $x^3 \to -\infty$.

Note that the function f_∞ only depends on a because translating (a, b) in the b-direction rescales both functions $\phi_\infty^{(i)}$ and leaves their ratio unchanged.

Proof. Consider the function $g(a) = \exp(\pi i f_\infty(a^3))$ defined on the sector

$$\Sigma = \{a \in \mathbb{C} : 0 < \arg(a) < \pi/3\}.$$

Note that g maps the two boundary rays of Σ to segments of circles, namely the circle of Appolonius r_+, and the unit circle $|z| = 1$. Thus again we can apply the usual proof of the Schwarz triangle theorem [9, §V.7]. We see that g extends analytically over the boundary of $\Sigma \subset \mathbb{C}$ and satisfies

$$T_\infty \circ g(1/a) = a^{3/2} \cdot h_\infty(a)$$

for $0 < |a| \ll 1$, where $h_\infty(a)$ is analytic near $a = 0$ and T_∞ is a Möbius transformation.

Consider the function $h(a) = \exp(\pi i a^{3/2})$. Then

$$\frac{h''(a)}{h'(a)} = \frac{3a^{3/2} + 1}{2a}.$$

So the Schwarzian is

$$\mathcal{S}(h) = \left(\frac{h''(a)}{h'(a)}\right)' - \frac{1}{2}\left(\frac{h''(a)}{h'(a)}\right)^2 = -\frac{9a^3 + 5}{8a^2}.$$

In particular, the Schwarzian $\mathcal{S}(h)$ has a simple pole at $a = \infty$.

Now standard arguments show that the Schwarzian derivative $\mathcal{S}(g)$ is holomorphic in Σ and extends to a meromorphic function on a neighbourhood of its boundary, with a pole at $a = \infty$. Moreover, $\mathcal{S}(g)$ takes the boundary of Σ to itself. It follows that $\mathcal{S}(g)(a) = \lambda \cdot a$ for some $\lambda \in \mathbb{R}$. By precomposing $f_\infty(t)$ with a rescaling we can reduce this to $\mathcal{S}(g)(a) = \frac{1}{3}a$. By general properties of the Schwarzian it then follows that $g(a)$ is given by a ratio of solutions of the linear differential equation

$$3y''(a) - ay(a) = 0,$$

a variant of the Airy equation. Since the solutions to this equation are precisely the functions $\phi_\infty^{(i)}(a, b)$ (as can easily by checked by differentiating under the integral sign), this completes the proof. □

Lemma 3.4. *In a neighbourhood of $t = \infty$ there is an expression*

$$T \circ g(t) = \exp(\pi i t^{1/2}) \cdot k(t)$$

where T is a Möbius transformation fixing $g(\infty) = 0$ and $k(t)$ is holomorphic at $t = 1$ and real-valued for t real.

Proof. The intervals $\mathbb{R}_{\geqslant 0}$ and $\mathbb{R}_{\leqslant 0}$ are mapped by g to two arcs of circles which meet at 0 at an angle of $\pi/2$. We take a Möbius transformation T which fixes 0 and maps these two arcs of circles to the rays $i \cdot \mathbb{R}_{\geqslant 0}$ and $\mathbb{R}_{\geqslant 0}$ respectively. Consider now the function

$$h(t) = \big(\frac{1}{i\pi} \log T(g(t))\big)^2,$$

defined by some branch of log near 0. Then h is holomorphic in the upper-half plane and extends continuously to a real-valued function on the intervals $(0,1)$ and $(1,\infty)$. The last thing to check is that it is continuous at $t=1$: given this, the reflection principle shows that h is holomorphic at $t=1$ and the usual argument can be applied. For the claim note that if $H(z)=\exp(\pi i z^{1/2})$ then $H^{-1}\circ T\circ H$ is holomorphic at $z=0$. Thus it is enough to consider $H^{-1}\circ g(t) = f(t)^2$. But this is continuous at $t=0$. □

4. Stability conditions

We let $\operatorname{Stab}_*(\mathrm{D}_n)$ denote the connected component of the space of stability conditions on D_n containing stability conditions whose heart is the canonical one. We set

$$\mathbb{P}\operatorname{Stab}_*(\mathrm{D}_n) = \operatorname{Stab}_*(\mathrm{D}_n)/\mathbb{C}.$$

It is a complex manifold locally modelled on the projective space

$$\mathbb{P}^1 = \mathbb{P}\operatorname{Hom}_{\mathbb{Z}}(K_0(\mathrm{D}_n),\mathbb{C}).$$

If σ is a stability condition we set $\S(\sigma)$ to be the set of indecomposable semistable objects of σ. We also set $\mathbb{P}\S(\sigma)$ to be the set of such objects up to shift. We now define an open subset $\mathcal{U}_n \subset \mathbb{P}\operatorname{Stab}_*(\mathrm{D}_n)$ as follows.

Definition 4.1. A projective stability condition $\bar{\sigma} \in \mathbb{P}\operatorname{Stab}_*(\mathrm{D}_n)$ lies in $\mathcal{U}_n$ if one of the following two conditions holds

(a) $\mathbb{P}\S(\sigma) = \{S_1, S_2\}$ and $0 < \phi(S_2) - \phi(S_1) < (n-2)/2$,

(b) $\mathbb{P}\S(\sigma) = \{S_1, S_2, E\}$ and

$$0 \leqslant \phi(S_1) - \phi(S_2) < \phi(E[1]) - \phi(S_1), \quad 0 \leqslant \phi(S_1) - \phi(S_2) < \phi(S_2[1]) - \phi(E). \tag{5}$$

The point of this definition is the following result.

Proposition 4.2. *For any $2 \leqslant n \leqslant \infty$ the domain $\mathcal{U}_n$ is a fundamental domain for the action of $\mathbb{P}\operatorname{Aut}_*(\mathrm{D}_n)$ on $\mathbb{P}\operatorname{Stab}_*(\mathrm{D}_n)$.*

Proof. Suppose a projective stability condition σ lies in the intersection $\mathcal{U}_n \cap \Phi^{-1}(\mathcal{U}_n) \subset \mathbb{P}\operatorname{Stab}_*(\mathrm{D}_n)$ for some $\Phi \in \operatorname{Aut}_*(\mathrm{D}_n)$. This means that $\sigma \in \mathcal{U}_n$ and also $\Phi(\bar{\sigma}) \in \mathcal{U}_n$. There are two cases to consider, corresponding to the two parts of Definition 4.1.

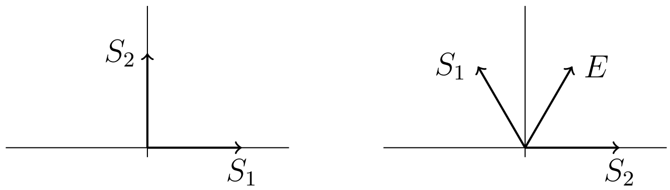

Figure 5. Stability conditions corresponding to orbifold points of $\mathbb{P}\operatorname{Stab}_*(\mathrm{D}_n)$

Suppose first that $\mathbb{P}\S(\sigma) = \{S_1, S_2\}$. Then Φ maps each S_i to an S_j up to shift. Given the Hom-spaces between S_1 and S_2 it is easy to see that if Φ defines a non-trivial element of $\mathbb{P}\operatorname{Aut}_*(\mathrm{D}_n)$ then we must have $n < \infty$ and

$$\Phi(S_1, S_2) = (S_2, S_1[n-2]))$$

up to shift. But then for $\Phi(\sigma)$ to lie in $\mathcal{U}_n$ we must have $n-2-(\phi(S_2)-\phi(S_1)) < (n-2)/2$ which gives a contradiction.

The second case is when $\mathbb{P}\S(\sigma) = \{S_1, S_2, E\}$. Then Φ preserves this set of objects up to shift. Given the maps between them, and using Lemma 2.2, it follows that $\Phi \in \mathbb{P}\operatorname{Aut}_*(\mathrm{D}_n)$ lies in the order 3 subgroup generated by Σ. Noticing that the inequalities (5) are equivalent to

$$|Z(S_2)| < |Z(E)|, \quad |Z(S_1)| < |Z(E)| \tag{6}$$

where Z is the central charge for σ, they give a contradiction.

Now consider the union of the closures of the regions $\Phi(\mathcal{U}_n)$ for $\Phi \in \mathbb{P}\operatorname{Aut}_*(\mathrm{D}_n)$. This subset of $\mathbb{P}\operatorname{Stab}_*(\mathrm{D}_n)$ is closed because it is a locally-finite union of closed subsets. To prove that it is open consider a stability condition σ defining a point in the boundary of $\mathcal{U}_n$. Again, there are two possibilities, corresponding to the two parts of Definition 4.1. In the first case, $\mathbb{P}\S(\sigma) = \{S_1, S_2\}$ and $\phi(S_2) - \phi(S_1) = (n-2)/2$. Then a neighbourhood of σ is covered by the closures of the regions $\mathcal{U}_n$ and $\Upsilon(\mathcal{U}_n)$. In the second case $\mathbb{P}\S(\sigma) = \{S_1, S_2, E\}$ and one or both of the two inequalities (5) is not strict. Then a neighbourhood of σ is covered by the closures of the regions $\mathcal{U}_n$, $\Sigma(\mathcal{U}_n)$ and $\Sigma^2(\mathcal{U}_n)$. This completes the proof. □

Remarks 4.3. (a) When $n < \infty$ there are two special points in the boundary of $\mathcal{U}_n$: one is fixed by Σ and the other by Υ. In the case $n = \infty$ only the order 3 point fixed by Σ exists. These projective stability conditions are illustrated in Figure 5.

(b) It follows from this result that an autoequivalence in $\operatorname{Aut}(\mathrm{D}_n)$ is reachable precisely if it preserves the connected component $\operatorname{Stab}_*(\mathrm{D}_n)$. Moreover by [11, Corollary 5.3]) such an autoequivalence acts trivially on $\operatorname{Stab}_*(\mathrm{D}_n)$ precisely if it acts trivially on the principal component $\mathrm{EG}^o(\mathrm{D}_n)$.

Proposition 4.4. *Let $2 \leqslant n \leqslant \infty$. Then the function*

$$g(\sigma) = \frac{1}{\pi i} \log \frac{Z(S_1)}{Z(S_2)}$$

defines a biholomorphic map between the regions $\mathcal{U}_n$ and R_n.

Proof. The region $\mathcal{U}_n$ consists of two parts, corresponding to conditions (a) and (b) of Definition 4.1. In the first part S_1 and S_2 are the only indecomposable semistable objects. This implies that $\phi(S_2) > \phi(S_1)$ since otherwise the extension E would also be semistable. Combined with the inequality in Definition 4.1 this gives

$$0 < \phi(S_2) - \phi(S_1) < (n-2)/2.$$

Any stability condition for which S_1 and S_2 are the only indecomposable semistable objects is clearly determined up to the $\mathbb{C}$-action by $\log Z(S_2)/Z(S_1)$, and it is also easy to see that any possible value compatible with the above constraint is possible. So the image of this part of $\mathcal{U}_n$ is precisely the strip $(2-n)/2 < \mathrm{Re}(z) < 0$.

In the second part of the region $\mathcal{U}_n$, all three objects S_1, S_2 and E are semistable. The existence of nonzero maps $S_1 \to S_2[1]$ implies the image of this part of $\mathcal{U}_n$ lies in the strip $0 \leqslant \mathrm{Re}(z) < 1$. Now the inequalities (5) in Definition 4.1 (or equivalently (6)) imply that this image is one third of this region, divided by the order 3 subgroup generated by Σ in $\mathrm{Aut}_*(\mathrm{D}_n)$. To see it is precisely the left part of R_n, we only need to notice that the boundaries $r_\pm$, defined by (4), of R_n correspond to the points where $|Z(S_1)+Z(S_2)| = |Z(E)| = |Z(S_1)|$ and $|Z(S_1)+Z(S_2)| = |Z(E)| = |Z(S_2)|$. □

We can now prove a projectivised version of Theorem 1.1. Recall that the quotient $\mathfrak{h}/W$ is isomorphic to $\mathbb{C}^2$ with co-ordinates (a,b) by setting

$$p(x) = (x-u_1)(x-u_2)(x-u_3) = x^3 + ax + b.$$

The image of the root hyperplanes $u_i = u_j$ is the discriminant

$$\Delta = \{(a,b) \in \mathbb{C}^2 : 4a^3 + 27b^2 = 0\}.$$

Note that $\mathfrak{h}$ has a natural $\mathbb{C}^*$ action rescaling the u_i co-ordinates with weight 1. This acts on (a,b) with weights $(2,3)$. We thus have

$$\mathbb{C}^* \backslash (\mathfrak{h} \setminus \{0\})/W \cong \mathbb{P}(2,3).$$

The weighted projective space $\mathbb{P}(2,3)$ contains two orbifold points with stabilizer groups μ_2 and μ_3 respectively. The image of the discriminant is a single (non-orbifold) point which we also label Δ.

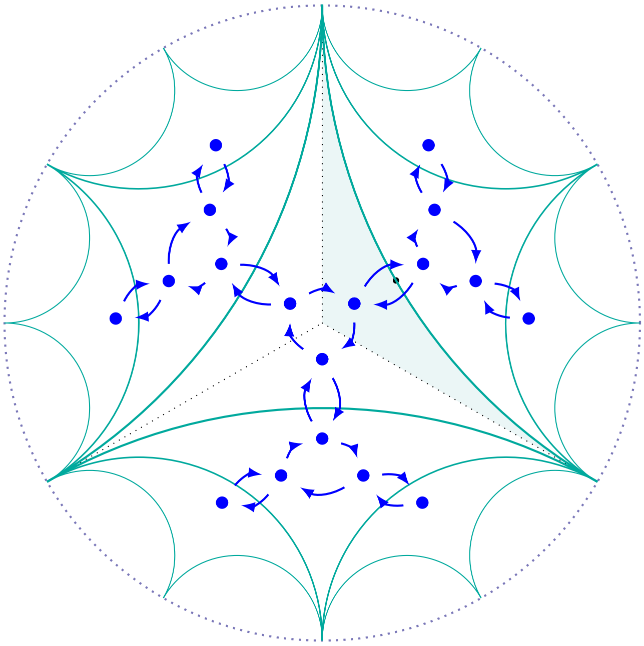

FIGURE 6. Exchange graph as the skeleton of space of stability conditions

In the $n = \infty$ case we consider the quotient of $\mathbb{P}^1$ by μ_3 given by $[1 : z] \mapsto [1 : e^{2\pi i/3} z]$. We label the two orbifold points $\{\mu_3, \infty\}$.

Theorem 4.5. *(a) For $2 \leqslant n < \infty$ the action of $\mathbb{P}\operatorname{Aut}_*(\mathrm{D}_n)$ on $\mathbb{P}\operatorname{Stab}_*(\mathrm{D}_n)$ is quasi-free and there is an isomorphism of complex orbifolds*

$$\mathbb{P}\operatorname{Stab}_*(\mathrm{D}_n)/\mathbb{P}\operatorname{Aut}_*(\mathrm{D}_n) \cong \mathbb{P}(2,3) \setminus \{\Delta\}.$$

(b) The action of $\mathbb{P}\operatorname{Aut}(\mathrm{D}_\infty)$ on $\mathbb{P}\operatorname{Stab}(\mathrm{D}_\infty)$ is quasi-free and there is an isomorphism of complex manifolds

$$\mathbb{P}\operatorname{Stab}(\mathrm{D}_\infty)/\mathbb{P}\operatorname{Aut}_*(\mathrm{D}_\infty) \cong \mathbb{C}/\mu_3.$$

Proof. We identify the upper half-plane arising in the last section with the upper half-plane in the coarse moduli space of the orbifold $\mathbb{P}(2,3)$, in such a way that the points $(0, 1, \infty)$ correspond to (μ_2, Δ, μ_3). Combining the map g of Prop. 4.4 and the inverse of the maps f_n of the last Section gives a biholomorphic map

$$\mathfrak{U}_n \xrightarrow{g} R_n \xrightarrow{f_n^{-1}} P.$$

Here, we view $\mathfrak{U}_n$ as an open dense subset of $\mathbb{P}\operatorname{Stab}_*(\mathrm{D}_n)/\operatorname{Aut}_*(\mathrm{D}_n)$, and $P = \mathbb{P}(2,3) \setminus ([\mu_2, \Delta] \cup [\Delta, \mu_3])$ is the union of two copies of the upper half-plane glued along the boundary component $[\mu_2, \mu_3]$.

By definition, the map g extends over the boundary of $\mathcal{U}_n$ and sends the two types of boundary points (corresponding to parts (a) and (b) of Definition 4.1) to the boundaries on the left and right of Figures 3 – 4 respectively. Under the map f_n^{-1} these boundaries become identified with $[\mu_2, \infty]$ and $[\mu_3, \infty]$ respectively. The result then follows.

The case $n = \infty$ proceeds along similar lines. We identify the upper half-plane with the upper half-plane in the coarse moduli space of the resulting orbifold $\mathbb{P}^1/\mu_3$. The composite $f_\infty^{-1} \circ g$ then identifies the dense open subset $\mathcal{U}_\infty$ of $\mathbb{P}\operatorname{Stab}(\mathrm{D}_\infty)/\mathbb{P}\operatorname{Aut}(\mathrm{D}_\infty)$ with the union of two copies of the upper half-plane glued along one of the boundary components $[\mu_3, \infty]$. The rest of the argument is then as above. □

Remark 4.6. Note that $\mathbb{P}\operatorname{Stab}(\mathrm{D}_\infty)$ is isomorphic to $\mathbb{C}$ and the action of $\mathbb{P}\operatorname{Aut}(\mathrm{D}_\infty) \cong \mu_3$ corresponds to the usual action of μ_3 on $\mathbb{C}$ by multiplication by a primitive third root of unity.

We can now lift Theorem 4.5 to obtain a proof of our main theorem.

Proof of Theorem 1.1. We have a diagram of complex manifolds and holomorphic maps

$$\begin{array}{ccc} \operatorname{Stab}_*(\mathrm{D}_n)/\operatorname{Sph}_*(\mathrm{D}_n) & & \mathbb{C}^2 \setminus \Delta \\ \downarrow & & \downarrow \\ \mathbb{P}\operatorname{Stab}_*(\mathrm{D}_n)/\mathbb{P}\operatorname{Sph}_*(\mathrm{D}_n) & \xrightarrow{\theta_n} & \mathbb{P}(2,3) \setminus \Delta \end{array}$$

The vertical arrows are $\mathbb{C}^*$-bundles, and the horizontal arrow θ_n is the isomorphism of Theorem 4.5. We would like to complete the diagram by filling in an upper horizontal isomorphism satisfying the property claimed in Theorem 1.1. Note that by construction the central charge map

$$\mathbb{P}\operatorname{Stab}(\mathrm{D}_n) \to \mathbb{P}^1$$

corresponds under the isomorphism θ_n to the multi-valued map given by ratios of the functions $\phi_n^{(i)}(a, b)$. These functions lift to $\mathbb{C}^2 \setminus \Delta$ and are then scaled with weight $3n-4 > 0$ by the $\mathbb{C}^*$-action. There is therefore a unique way to fill in the upper arrow so that the multi-valued central charge map on $\operatorname{Stab}(\mathrm{D}_n)$ corresponds to $\phi_n^{(i)}(a, b)$.

In the $n = \infty$ we have a similar diagram

$$\begin{array}{ccc} \operatorname{Stab}_*(\mathrm{D}_\infty) & & \mathbb{C}^2 \\ \downarrow & & \downarrow \\ \mathbb{P}\operatorname{Stab}_*(\mathrm{D}_\infty) & \xrightarrow{\theta_\infty} & \mathbb{C} \end{array}$$

in which the vertical arrows are $\mathbb{C}$-bundles. The bundle on the right is just the projection $\mathbb{C}^2 \to \mathbb{C}$ given by $(a,b) \mapsto a$. By construction the central charge map

$$\mathbb{P}\,\mathrm{Stab}_*(\mathrm{D}_\infty) \to \mathbb{P}^1$$

is given by ratios of the functions $\phi_\infty^{(i)}(a,b)$. These functions lift to $\mathbb{C}^2$ and are then scaled by weight e^b by translation by $(0,b)$. There is therefore a unique way to fill in the upper arrow so that the central charge map on $\mathrm{Stab}(\mathrm{D}_\infty)$ corresponds to $\phi_\infty^{(i)}(a,b)$.

Remark 4.7. In the $n=\infty$ case the auto-equivalence group is $\mathbb{Z}$ generated by Σ. The induced action on $\mathfrak{h}/W$ is given by $(a,b) \mapsto (e^{2\pi i/3}a, b+\pi i/3)$. The element $\Sigma^3=[1]$ then fixes a and acts by $b \mapsto b+\pi i$.

Appendix A. Hypergeometric equation for the twisted periods

In this section we prove that the twisted periods satisfy the hypergeometric differential equation (**??**) appearing in the proof of Theorem 3.1.

Let us fix $a \in \mathbb{C}$ and consider the function

$$f_a(h) = h^{-(\nu+1)} \int e^{h(x^3+ax)} dx.$$

Setting $t = h^{1/3} \cdot x$ we see that

$$f_a(h) = h^{-(\nu+\frac{4}{3})} \int e^{t^3+h^{2/3}at} dt.$$

Introduce the differential operator

$$D_h = h\partial_h + \nu + 1.$$

Then

$$(D_h + \frac{1}{3}) f_a(h) = \frac{2}{3} \cdot h^{-(\nu+\frac{4}{3})} \int h^{\frac{2}{3}} \cdot at \cdot e^{t^3+h^{2/3}at} dt.$$

Repeating we obtain

$$(D_h - \frac{1}{3})(D_h + \frac{1}{3}) f_a(h) = \frac{4}{9} \cdot h^{-(\nu+\frac{4}{3})} \int h^{\frac{4}{3}} \cdot (at)^2 \cdot e^{t^3+h^{2/3}at} dt,$$

and it follows that

$$\left((D_h - \frac{1}{3})(D_h + \frac{1}{3}) + \frac{4a^3}{27} \cdot h^2 \right) f(a,h) =$$

$$= h^{-(\nu+\frac{4}{3})} \cdot h^{\frac{4}{3}} \cdot \frac{4a^2}{27} \int (3t^2 + h^{\frac{2}{3}} a) \cdot e^{t^3+h^{2/3}at}\, dt = 0.$$

Now consider the (inverse) Laplace transform

$$p_a(b) = \int e^{bh} f_a(h)\, dh = \int\int e^{h(x^3+ax+b)} \cdot h^{-(\nu+1)}\, dx\, dh.$$

Exchanging the order of integration and using

$$\int e^{h(y+b)} \cdot h^{-(\nu+1)} dh = (-\nu-1)! \cdot (y+b)^{\nu},$$

valid for $\mathrm{Re}(\nu) > -1$ this becomes

$$p_a(b) = (-\nu-1)! \cdot \int (x^3+ax+b)^{\nu}\, dx.$$

Under the inverse transform $h\partial_h$ becomes $-b\partial_b - 1$ so the transform of the operator D_h is

$$M_b = (-b\partial_b + \nu).$$

The twisted periods therefore satisfy the differential equation

$$\left((-b\partial_b + \nu + \frac{1}{3})(-b\partial_b + \nu - \frac{1}{3}) + \frac{4a^3}{27} \cdot \partial_b^2\right) p_a(b) = 0$$

which can be rewritten

$$\left(\frac{4a^3}{27} + b^2\right)\partial_b^2 + (1+\alpha+\beta)b\partial_b + \alpha\beta = 0 \tag{7}$$

with $\alpha = 1/3 - \nu$ and $\beta = -1/3 - \nu$. Our derivation holds for $\mathrm{Re}(\nu) > -1$ but by analytic continuation the result holds in general.

The differential equation (7) is second order in b with three regular singularities. To put it in hypergeometric form substitute

$$b = d(2z-1) \text{ with } 4a^3 + 27d^2 = 0$$

so that the singularities lie at $z \in \{0, 1, \infty\}$. The equation now becomes

$$z(1-z)\partial_z^2 + \big(\gamma - (1+\alpha+\beta)z\big)\partial_z - \alpha\beta = 0,$$

with $\gamma = 1/2 - \nu$. Setting $\nu = (n-2)/2$ gives the claim.

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