EQUIVALENCES OF TRIANGULATED CATEGORIES AND FOURIER–MUKAI TRANSFORMS

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Abstract

We give a condition for an exact functor between triangulated categories to be an equivalence. Applications to Fourier–Mukai transforms are discussed. In particular, we obtain a large number of such transforms for K3 surfaces.

1. *Introduction*

Let X and Y be smooth projective varieties of the same dimension, and let $\mathscr{P}$ be a vector bundle on $X\times Y$. Define a functor

$$F: D(Y) \longrightarrow D(X)$$

between the derived categories of sheaves on Y and X by the formula

$$F(-) = \mathbf{R}\pi_{X,*}(\mathscr{P}\otimes\pi_Y^*(-)),$$

where $X \xleftarrow{\pi_X} X\times Y \xrightarrow{\pi_Y} Y$ are the projection maps. Functors of this type which are equivalences of categories are called *Fourier–Mukai transforms*, and have proved to be powerful tools for studying moduli spaces of vector bundles [**4**, **5**, **11**].

A vector bundle $\mathscr{P}$ on $X\times Y$ is called *strongly simple* over Y if, for each point $y\in Y$, the bundle $\mathscr{P}_y$ on X is simple, and if, for any two distinct points y_1, y_2 of Y, and any integer i, one has

$$\operatorname{Ext}^i_X(\mathscr{P}_{y_1},\mathscr{P}_{y_2}) = 0.$$

One might think of the family $\{\mathscr{P}_y : y\in Y\}$ as an 'orthonormal' set of bundles on X.

The following basic result allows one to construct many examples of Fourier–Mukai transforms.

Theorem 1.1. *The functor F is fully faithful if and only if $\mathscr{P}$ is strongly simple over Y. It is an equivalence of categories precisely when one also has $\mathscr{P}_y = \mathscr{P}_y\otimes\omega_X$ for all $y\in Y$.*

The first statement is well known [**3**, **8**], but the second part is new. In this paper we shall prove Theorem 1.1, along with some more general results concerning exact functors between triangulated categories.

As an example of the use of Theorem 1.1, we have the following.

Corollary 1.2. *Let X be an algebraic K3 surface, and let Y be a fine, compact, 2-dimensional moduli space of stable vector bundles on X. Then Y is also a K3 surface, and if $\mathscr{P}$ is a universal bundle on $X\times Y$, then the functor F is an equivalence of categories.*

Received 4 July 1997; revised 8 October 1997.

1991 *Mathematics Subject Classification* 18E30, 14J28.

Proof. The fact that Y is a K3 surface is Theorem 1.4 of [**12**]. Since ω_X is trivial, it is enough to check that $\mathscr{P}$ is strongly simple over Y. This follows from [**12**, Proposition 3.12], because any stable sheaf which moves in a 2-dimensional moduli is semi-rigid.

NOTATION. All our schemes will be Noetherian schemes. A sheaf on a scheme X will mean a coherent $\mathscr{O}_X$-module, and a point of X will mean a closed point. If x is a point of X, then $\mathscr{O}_x$ denotes the structure sheaf of x with reduced scheme structure.

If a and b are objects of a triangulated category $\mathscr{A}$, put

$$\mathrm{Hom}^i_{\mathscr{A}}(a,b) = \mathrm{Hom}_{\mathscr{A}}(a, T^i b),$$

where $T\colon \mathscr{A} \to \mathscr{A}$ is the translation functor.

If X is a scheme, then $D(X)$ will denote the bounded derived category of sheaves on X. For an object E of $D(X)$, let

$$E^\vee = \mathbf{R}\,\mathrm{Hom}_{\mathscr{O}_X}(E, \mathscr{O}_X).$$

We shall write $H^i(E)$ for the ith cohomology sheaf of E, and $E[n]$ for the object obtained by shifting E to the left by n places. We say that E is a sheaf if $H^i(E) = 0$ when $i \neq 0$.

If $f\colon X \to Y$ is a morphism of schemes, and E is an object of $D(Y)$, then $\mathbf{L}_p f^*(E)$ denotes the $(-p)$th cohomology object of $\mathbf{L}f^*(E)$.

ACKNOWLEDGEMENTS. I learnt a great deal from the papers of Bondal and Orlov. I am also grateful to my supervisor, Antony Maciocia, for all his help and encouragement. Finally, I should like to thank the referee, who pointed out a major error in an earlier version of the paper.

2. *Fully faithful functors*

In this section we give a general criterion for an exact functor between triangulated categories to be fully faithful. Its proof is very similar to that of [**14**, Lemma 2.15].

DEFINITION 2.1. Let $\mathscr{A}$ be a triangulated category. A subclass Ω of the objects of $\mathscr{A}$ will be called a *spanning class* for $\mathscr{A}$ if, for any object a of $\mathscr{A}$,

$$\mathrm{Hom}^i_{\mathscr{A}}(\omega, a) = 0 \quad \forall \omega \in \Omega \quad \forall i \in \mathbb{Z} \quad \Rightarrow \quad a \cong 0,$$
$$\mathrm{Hom}^i_{\mathscr{A}}(a, \omega) = 0 \quad \forall \omega \in \Omega \quad \forall i \in \mathbb{Z} \quad \Rightarrow \quad a \cong 0.$$

EXAMPLE 2.2. If X is a smooth projective variety, then the set

$$\Omega = \{\mathscr{O}_x : x \in X\}$$

is a spanning class for $\mathscr{A} = D(X)$.

Proof. For any object a of $\mathscr{A} = D(X)$, and any $x \in X$, there is a spectral sequence

$$E_2^{p,q} = \mathrm{Ext}^p_X(H^{-q}(a), \mathscr{O}_x) \quad \Rightarrow \quad \mathrm{Hom}^{p+q}_{\mathscr{A}}(a, \mathscr{O}_x).$$

If a is non-zero, let q_0 be the maximal value of q such that $H^q(a)$ is non-zero, and assume that x is a closed point in the support of $H^{q_0}(a)$. Then there is a non-zero element of $E_2^{0,-q_0}$ which survives to give an element of $\mathrm{Hom}_{\mathscr{A}}^{-q_0}(a, \mathscr{O}_x)$. Serre duality then gives a non-zero element of $\mathrm{Hom}^i_{\mathscr{A}}(\mathscr{O}_x, a)$, where $i = \dim X + q_0$.

THEOREM 2.3. *Let $\mathscr{A}$ and $\mathscr{B}$ be triangulated categories, and let $F:\mathscr{A} \to \mathscr{B}$ be an exact functor with a left and a right adjoint. Then F is fully faithful if and only if there exists a spanning class Ω for $\mathscr{A}$ such that for all elements ω_1, ω_2 of Ω, and all integers i, the homomorphism*

$$F:\mathrm{Hom}^i_{\mathscr{A}}(\omega_1, \omega_2) \longrightarrow \mathrm{Hom}^i_{\mathscr{A}}(F\omega_1, F\omega_2)$$

is an isomorphism.

Proof. One implication is clear, so let us assume the existence of Ω and prove that F is fully faithful.

Let $H:\mathscr{B} \to \mathscr{A}$ be a right adjoint of F, and let

$$\eta: 1_{\mathscr{A}} \longrightarrow H \circ F, \quad \varepsilon: F \circ H \longrightarrow 1_{\mathscr{B}},$$

be the unit and counit, respectively, of the adjunction $F \dashv H$. Similarly, let $G:\mathscr{B} \to \mathscr{A}$ be a left adjoint of F, and let

$$\zeta: 1_{\mathscr{B}} \longrightarrow F \circ G, \quad \delta: G \circ F \longrightarrow 1_{\mathscr{A}},$$

be the unit and counit of $G \dashv F$. Note that by [**14**, Lemma 1.2], G and H are also exact functors.

For any pair of objects a and b of $\mathscr{A}$, and any integer i, there is a commutative diagram of group homomorphisms

$$\begin{array}{ccc} \mathrm{Hom}^i_{\mathscr{A}}(a, b) & \xrightarrow{\eta(b)_*} & \mathrm{Hom}^i_{\mathscr{B}}(a, HFb) \\ {\scriptstyle \delta(a)^*}\downarrow & & \downarrow{\scriptstyle \beta} \\ \mathrm{Hom}^i_{\mathscr{A}}(GFa, b) & \xrightarrow{\alpha} & \mathrm{Hom}^i_{\mathscr{B}}(Fa, Fb) \end{array}$$

in which $\alpha = \zeta(Fa)^* \circ F$ and $\beta = \varepsilon(Fb)_* \circ F$ are isomorphisms, and the common diagonal is the map

$$F:\mathrm{Hom}^i_{\mathscr{A}}(a, b) \longrightarrow \mathrm{Hom}^i_{\mathscr{B}}(Fa, Fb).$$

When a and b are elements of Ω, this map is an isomorphism (by hypothesis), so all the maps in (1) are isomorphisms.

First we show that for any object a in Ω, the morphism $\delta(a)$ is an isomorphism. To see this, embed $\delta(a)$ in a triangle of $\mathscr{A}$:

$$GFa \xrightarrow{\delta(a)} a \longrightarrow c \longrightarrow T(GFa).$$

For any object b of Ω, we can apply the functor $\mathrm{Hom}_{\mathscr{A}}(-, b)$ to this triangle, and obtain a long exact sequence of groups

$$\cdots \longleftarrow \mathrm{Hom}_{\mathscr{A}}(GFa, b) \xleftarrow{\delta(a)^*} \mathrm{Hom}_{\mathscr{A}}(a, b) \longleftarrow \mathrm{Hom}_{\mathscr{A}}(c, b) \longleftarrow$$
$$\longleftarrow \mathrm{Hom}^{-1}_{\mathscr{A}}(GFa, b) \xleftarrow{\delta(a)^*} \mathrm{Hom}^{-1}_{\mathscr{A}}(a, b) \longleftarrow \cdots$$

But since the maps $\delta(a)^*$ are all isomorphisms, this implies that

$$\mathrm{Hom}^i_{\mathscr{A}}(c,b)=0 \quad \forall b\in\Omega \quad \forall i\in\mathbb{Z},$$

so $c\cong 0$ and $\delta(a)$ is an isomorphism.

Now take an object b of $\mathscr{A}$, embed the morphism $\eta(b)$ in a triangle

$$b\xrightarrow{\eta(b)} HFb\longrightarrow c\longrightarrow Tb,$$

and apply the functor $\mathrm{Hom}_{\mathscr{A}}(a,-)$ with $a\in\Omega$. The homomorphisms

$$\eta(a)_*:\mathrm{Hom}^i_{\mathscr{A}}(a,b)\longrightarrow \mathrm{Hom}^i_{\mathscr{A}}(a,HFb)$$

appearing in the resulting long exact sequence are isomorphisms, because of the commuting diagram (1) and the fact that $\delta(a)^*$ is an isomorphism. Arguing as above, we conclude that $c\cong 0$ and hence that $\eta(b)$ is an isomorphism. Since b was arbitrary, this is enough to show that F is fully faithful.

3. *Equivalences of triangulated categories*

Here we give a condition for a fully faithful exact functor between triangulated categories to be an equivalence. We refer to [**9**, VIII.2] for the notion of biproducts in an additive category.

DEFINITION 3.1. A triangulated category $\mathscr{A}$ will be called *indecomposable* if, whenever $\mathscr{A}_1$ and $\mathscr{A}_2$ are full subcategories of $\mathscr{A}$ satisfying

(a) for every object a of $\mathscr{A}$ there exist objects $a_j\in\mathrm{Ob}(\mathscr{A}_j)$ such that a is a biproduct of a_1 and a_2,

(b) for any pair of objects $a_j\in\mathrm{Ob}(\mathscr{A}_j)$,

$$\mathrm{Hom}^i_{\mathscr{A}}(a_1,a_2)=\mathrm{Hom}^i_{\mathscr{A}}(a_2,a_1)=0 \quad \forall i\in\mathbb{Z},$$

there exists j such that $a\cong 0$ for all $a\in\mathrm{Ob}(\mathscr{A}_j)$.

EXAMPLE 3.2. If X is a scheme, then $D(X)$ is indecomposable if and only if X is connected.

Proof. We suppose that X is connected, and prove that $\mathscr{A}=D(X)$ is indecomposable. The (easy) converse is left to the reader.

Suppose that $\mathscr{A}_1$ and $\mathscr{A}_2$ are full subcategories of $\mathscr{A}$ satisfying conditions (a) and (b) of Definition 3.1. For any integral closed subscheme Y of X, the sheaf $\mathscr{O}_Y$ is indecomposable, and is therefore isomorphic to some object of $\mathscr{A}_j$, for $j=1$ or 2. For any point $y\in Y$, we must then have that $\mathscr{O}_y$ is also isomorphic to an object of $\mathscr{A}_j$, since otherwise (b) would imply that $\mathrm{Hom}_{\mathscr{A}}(\mathscr{O}_Y,\mathscr{O}_y)=0$, which is not the case.

Let X_j be the union of those Y such that $\mathscr{O}_Y$ is isomorphic to an object of $\mathscr{A}_j$. Then X_1 and X_2 are closed subsets of X, and $X=X_1\cup X_2$. If a point $x\in X$ lies in X_1 and X_2, then $\mathscr{O}_x$ is isomorphic to an object of $\mathscr{A}_1$ and to an object of $\mathscr{A}_2$. This contradicts (b). Thus the union is disjoint, and the fact that X is connected implies that one of the X_j (without loss of generality, X_2) is empty. But then (b) implies that for any object a of $\mathscr{A}_2$ one has

$$\mathrm{Hom}^i_{\mathscr{A}}(a,\mathscr{O}_x)=0 \quad \forall i\in\mathbb{Z} \quad \forall x\in X,$$

and hence, by the argument of Example 2.2, $a\cong 0$. This completes the proof.

THEOREM 3.3. *Let $\mathscr{A}$ and $\mathscr{B}$ be triangulated categories, and let $F:\mathscr{A}\to\mathscr{B}$ be a fully faithful exact functor. Suppose that $\mathscr{B}$ is indecomposable, and that not every object of $\mathscr{A}$ is isomorphic to* 0. *Then F is an equivalence of categories if and only if F has a left adjoint G and a right adjoint H such that for any object b of $\mathscr{B}$,*

$$Hb \cong 0 \quad \Rightarrow \quad Gb \cong 0.$$

Proof. If F is an equivalence, then any quasi-inverse of F is a left and right adjoint for F.

For the converse, take an object b of $\mathscr{B}$ and (with notation as in Theorem 2.3) embed the morphism $\varepsilon(b)$ in a triangle of $\mathscr{B}$:

$$FHb \xrightarrow{\varepsilon(b)} b \longrightarrow c \longrightarrow T(FHb).$$

Applying H, one sees that $Hc \cong 0$, because the fact that F is fully faithful implies that the morphism $H(\varepsilon(b))$ is an isomorphism. Define full subcategories $\mathscr{B}_1$ and $\mathscr{B}_2$ of $\mathscr{B}$ consisting of objects satisfying $FHb \cong b$ and $Hb \cong 0$ respectively. Now our hypothesis implies that

$$\mathrm{Hom}^i_{\mathscr{B}}(b_1, b_2) = \mathrm{Hom}^i_{\mathscr{B}}(b_2, b_1) = 0 \quad \forall i \in \mathbb{Z},$$

whenever $b_j \in \mathscr{B}_j$. Furthermore, the lemma below applied to the triangle above shows that every object of $\mathscr{B}$ is a biproduct $b_1 \oplus b_2$. Since $\mathscr{B}$ is indecomposable, we must have

$$Hc \cong 0 \quad \Rightarrow \quad c \cong 0,$$

for any $c \in \mathrm{Ob}(\mathscr{B})$, so the morphism $\varepsilon(b)$ appearing above is an isomorphism. Since b was arbitrary, $F \circ H \cong 1_{\mathscr{B}}$ and F is an equivalence.

LEMMA 3.4. *Let $\mathscr{A}$ be a triangulated category, and let*

$$a_1 \xrightarrow{i_1} b \xrightarrow{p_2} a_2 \xrightarrow{0} Ta_1,$$

be a triangle of $\mathscr{A}$. Then b is a biproduct of a_1 and a_2 in $\mathscr{A}$.

Proof. Applying the functors $\mathrm{Hom}_{\mathscr{A}}(-, a_1)$ and $\mathrm{Hom}_{\mathscr{A}}(a_2, -)$, one obtains morphisms $p_1 : b \to a_1$ and $i_2 : a_2 \to b$, such that $p_1 \circ i_1 = 1_{a_1}$ and $p_2 \circ i_2 = 1_{a_2}$. The composition $p_2 \circ i_1$ is always 0, and replacing i_2 by $i_2 - i_1 \circ p_1 \circ i_2$, we can assume that $p_1 \circ i_2 = 0$. Then (see [**9**, VIII.2]) it is enough to check that the endomorphism of b given by

$$\phi = 1_b - i_1 \circ p_1 - i_2 \circ p_2$$

is the zero map. But this follows from the fact that $p_1 \circ \phi = p_2 \circ \phi = 0$.

4. *Integral functors*

Throughout this section, X and Y are smooth projective varieties over an algebraically closed field k of characteristic zero, and $\mathscr{P}$ is an object of $D(X \times Y)$. F denotes the exact functor

$$\Phi^{\mathscr{P}}_{Y\to X} : D(Y) \longrightarrow D(X)$$

defined by the formula

$$\Phi^{\mathscr{P}}_{Y\to X}(-) = \mathbf{R}\pi_{X,*}(\mathscr{P} \overset{\mathbf{L}}{\otimes} \pi_Y^*(-)).$$

Following Mukai, we call F an *integral functor*. Here we derive various general properties of such functors. Most of these appeared in some form in the original papers of Mukai on Abelian varieties [**10**, **11**].

Given a scheme S, one can define a relative version of F over S. This is the functor

$$F_S\colon D(S\times Y)\longrightarrow D(S\times X),$$

given by the formula

$$F_S(-)=\mathbf{R}\pi_{S\times X,*}(\mathscr{P}_S\overset{\mathbf{L}}{\otimes}\pi^*_{S\times Y}(-)),$$

where $S\times X\xleftarrow{\pi_{S\times X}}S\times X\times Y\xrightarrow{\pi_{S\times Y}}S\times Y$ are the projection maps, and $\mathscr{P}_S$ is the pull-back of $\mathscr{P}$ to $S\times X\times Y$.

The following result is similar to [**11**, Proposition 1.3].

LEMMA 4.1. *Let $g\colon T\to S$ be a morphism of schemes, and let E be an object of $D(S\times Y)$, of finite tor-dimension over S. Then there is an isomorphism*

$$F_T\circ\mathbf{L}(g\times 1_Y)^*(E)\cong\mathbf{L}(g\times 1_X)^*\circ F_S(E).$$

Proof. One needs to base-change around the following diagram.

$$\begin{array}{ccc} T\times X\times Y & \xrightarrow{(g\times 1_{X\times Y})} & S\times X\times Y \\ \Big\downarrow{\scriptstyle \pi_{T\times X}} & & \Big\downarrow{\scriptstyle \pi_{S\times X}} \\ T\times X & \xrightarrow{(g\times 1_X)} & S\times X \end{array}$$

This is justified by the same argument used to prove [**3**, Lemma 1.3].

We can now show that integral functors preserve families of sheaves. It is this property which makes integral functors useful for studying moduli problems; see also [**11**, Theorem 1.6].

PROPOSITION 4.2. *Let S be a scheme, and let $\mathscr{E}$ be a sheaf on $S\times Y$, flat over S. Suppose that for each $s\in S$, $F(\mathscr{E}_s)$ is a sheaf on X. Then there is a sheaf $\hat{\mathscr{E}}$ on $S\times X$, flat over S, such that for every $s\in S$, $\hat{\mathscr{E}}_s=F(\mathscr{E}_s)$.*

Proof. Let $\hat{\mathscr{E}}=F_S(\mathscr{E})$, and take a point $s\in S$. Applying Lemma 4.1 with $T=\{s\}$, we see that the derived restriction of $\hat{\mathscr{E}}$ to the fibre $\{s\}\times X$ is just $F(\mathscr{E}_s)$. The following lemma then shows that $\hat{\mathscr{E}}$ is a sheaf on $S\times X$, flat over S.

LEMMA 4.3. *Let $\pi\colon S\to T$ be a morphism of schemes, and for each point $t\in T$, let $i_t\colon S_t\to S$ denote the inclusion of the fibre $\pi^{-1}(t)$. Let $\mathscr{E}$ be an object of $D(S)$ such that for all $t\in T$, $\mathbf{L}i_t^*(\mathscr{E})$ is a sheaf on S_t. Then $\mathscr{E}$ is a sheaf on S, flat over T.*

Proof. For each point $t\in T$, consider the hypercohomology spectral sequence

$$E_2^{p,q}=\mathbf{L}_{-p}\,i_t^*(H^q(\mathscr{E}))\quad\Rightarrow\quad\mathbf{L}_{-(p+q)}\,i_t^*(\mathscr{E}).$$

By assumption, the right-hand side is zero unless $p+q=0$. If q_0 is the largest q such that $H^q(\mathscr{E})\neq 0$, then E_2^{0,q_0} survives in the spectral sequence for some $t\in T$, so $q_0=0$.

Now $H^0(\mathscr{E})$ must be flat over T, since otherwise $E_2^{-1,0}$ would survive for some $t \in T$. Finally, suppose $H^q(\mathscr{E}) \neq 0$ for some $q < 0$. Then we can find a largest such q, and this gives an element of $E_2^{0,q}$ which survives. Hence $H^q(\mathscr{E}) = 0$ unless $q = 0$ and $\mathscr{E}$ is a sheaf, flat over T.

In the next section we shall need the following.

LEMMA 4.4. *Suppose that $\mathscr{P}$ is a sheaf on $X \times Y$, flat over Y, and fix a point $y \in Y$. Then the homomorphism*

$$F\colon \operatorname{Ext}^1_Y(\mathscr{O}_y, \mathscr{O}_y) \longrightarrow \operatorname{Ext}^1_X(\mathscr{P}_y, \mathscr{P}_y) \tag{2}$$

is the Kodaira–Spencer map for the family $\mathscr{P}$ at the point y, if we identify the first space with the tangent space to Y at y in the usual way.

Proof. Let $D = \operatorname{Spec} k[\varepsilon]/\varepsilon^2$ denote the double point. We identify the tangent space $\mathrm{T}_y Y$ to Y at y with the set of morphisms $D \to Y$, taking the closed point of D to y. Given such a morphism f, we can pull $\mathscr{P}$ back, and obtain a deformation of the sheaf $\mathscr{P}_y$ on X, with base D. The set of such deformations is identified with $\operatorname{Ext}^1_X(\mathscr{P}_y, \mathscr{P}_y)$, and the Kodaira–Spencer map is the resulting linear map

$$\mathrm{T}_y Y \longrightarrow \operatorname{Ext}^1_X(\mathscr{P}_y, \mathscr{P}_y).$$

Returning to our homomorphism (2), note that we can identify the domain with the set of deformations of $\mathscr{O}_y$ over D, and the image with the set of deformations of $\mathscr{P}_y$ over D. If we do this, then it is easy to see that the map F is given just by applying the functor F_D.

Given an element $f\colon D \to Y$ of $\mathrm{T}_y Y$, the corresponding deformation of $\mathscr{O}_y$ over D is obtained by pulling back the family $\mathscr{O}_\Delta$ on $Y \times Y$ to $D \times Y$ using f (here Δ denotes the diagonal in $Y \times Y$). By Lemma 4.1, if we then apply F_D, we obtain the same result as if we first applied F_Y, which gives the sheaf $\mathscr{P}$ on $X \times Y$, and then pulled back via f. But this is the Kodaira–Spencer map for the family $\mathscr{P}$.

The following result is well known. Its proof is a straightforward application of Grothendieck–Verdier duality (see [**8**, Proposition 3.1] or [**3**, Lemma 1.2]).

LEMMA 4.5. *The functors*

$$G = \Phi_{X \to Y}^{\mathscr{P}^\vee \otimes \pi_X^* \omega_X[\dim X]}, \quad H = \Phi_{X \to Y}^{\mathscr{P}^\vee \otimes \pi_Y^* \omega_Y[\dim Y]},$$

are left and right adjoints for F, respectively.

5. *Applications to Fourier–Mukai transforms*

As in the last section, we fix smooth projective varieties X and Y over an algebraically closed field k of characteristic zero, and an object $\mathscr{P}$ of $D(X \times Y)$. F denotes the corresponding functor $\Phi_{Y \to X}^{\mathscr{P}}$. The following theorem was first proved by A. I. Bondal and D. O. Orlov, using ideas of Mukai.

THEOREM 5.1 ([**3**]). *The functor F is fully faithful if and only if for each point $y \in Y$,*

$$\mathrm{Hom}_{D(X)}(F\mathcal{O}_y, F\mathcal{O}_y) = k,$$

and for each pair of points $y_1, y_2 \in Y$, and each integer i,

$$\mathrm{Hom}^i_{D(X)}(F\mathcal{O}_{y_1}, F\mathcal{O}_{y_2}) = 0 \quad \textit{unless } y_1 = y_2 \textit{ and } 0 \leqslant i \leqslant \dim Y.$$

Proof. We must show that for any point y of Y, and any integer i, the homomorphism

$$F\colon \mathrm{Hom}^i_{D(Y)}(\mathcal{O}_y, \mathcal{O}_y) \longrightarrow \mathrm{Hom}^i_{D(X)}(F\mathcal{O}_y, F\mathcal{O}_y)$$

is an isomorphism. Theorem 2.3 will then give the result. By the commutative diagram (1), it will be enough to show that $\delta(\mathcal{O}_y)$ is an isomorphism. In fact, it will be enough to show that $GF\mathcal{O}_y \cong \mathcal{O}_y$, because then $\delta(\mathcal{O}_y)$ must be either an isomorphism or zero, and the latter is impossible, because $F(\delta(\mathcal{O}_y))$ has a left-inverse.

For any point $z \overset{i_z}{\hookrightarrow} Y$, there are isomorphisms of vector spaces

$$\mathbf{L}_p\, i_z^*(GF\mathcal{O}_y) \cong \mathrm{Hom}^p_{D(Y)}(GF\mathcal{O}_y, \mathcal{O}_z) \cong \mathrm{Hom}^p_{D(X)}(F\mathcal{O}_y, F\mathcal{O}_z)$$

coming from the adjunctions $i_z^* \dashv i_{z,*}$ [**7**, Corollary 5.11], and $G \dashv F$. Thus, by [**3**, Proposition 1.5], $GF\mathcal{O}_y$ is a sheaf supported at the point y. Furthermore, there is a unique morphism $GF\mathcal{O}_y \to \mathcal{O}_y$. If K is the kernel of this morphism, then one has a short exact sequence

$$0 \longrightarrow K \longrightarrow GF\mathcal{O}_y \xrightarrow{\delta(\mathcal{O}_y)} \mathcal{O}_y \longrightarrow 0,$$

and we must show that $K = 0$. Applying the functor $\mathrm{Hom}_{D(Y)}(-, \mathcal{O}_y)$, and using the diagram (1), it will be enough to show that the homomorphism

$$F\colon \mathrm{Hom}^1_{D(Y)}(\mathcal{O}_y, \mathcal{O}_y) \longrightarrow \mathrm{Hom}^1_{D(X)}(F\mathcal{O}_y, F\mathcal{O}_y) \tag{3}$$

is injective.

By [**10**, Proposition 1.3], $GF = \Phi^{\mathcal{Q}}_{Y \to Y}$ for some object $\mathcal{Q}$ of $D(Y \times Y)$. Since $GF\mathcal{O}_y$ is a sheaf for all $y \in Y$, Lemma 4.3 shows that $\mathcal{Q}$ is in fact a sheaf, flat over Y. Furthermore, by Lemma 4.4, the map

$$GF\colon \mathrm{Hom}^1_{D(Y)}(\mathcal{O}_y, \mathcal{O}_y) \longrightarrow \mathrm{Hom}^1_{D(Y)}(GF\mathcal{O}_y, GF\mathcal{O}_y)$$

is given by the Kodaira–Spencer map for the family $\mathcal{Q}$ at the point y. The following two lemmas show that for general $y \in Y$ this map is injective. Clearly, the map (3) must also be injective.

LEMMA 5.2. *Let Y be a projective variety over k, and let Q be a sheaf on Y supported at a point $y \in Y$. Suppose that*

$$\mathrm{Hom}_Y(Q, \mathcal{O}_y) = k.$$

Then Q is the structure sheaf of a zero-dimensional closed subscheme of Y.

Proof. There exists a short exact sequence

$$0 \longrightarrow P \longrightarrow Q \xrightarrow{g} \mathcal{O}_y \longrightarrow 0.$$

Suppose that $f\colon \mathcal{O}_Y \to Q$ is a non-surjective morphism of sheaves. Considering the cokernel of f shows that there is a non-zero morphism $h\colon Q \to \mathcal{O}_y$ such that $h \circ f = 0$. But, by hypothesis, h must be a multiple of g, so one must have $g \circ f = 0$, hence f comes from a morphism $\mathcal{O}_Y \to P$. Now

$$\dim_k H^0(Y, P) = \chi(P) < \chi(Q) = \dim_k H^0(Y, Q),$$

so there must be a morphism $\mathcal{O}_Y \to Q$ which is surjective.

LEMMA 5.3. *Let S and Y be varieties over k, with Y projective. Let $\mathcal{Q}$ be a sheaf on $S \times Y$, flat over S, such that for each $s \in S$, $\mathcal{Q}_s$ is the structure sheaf of a zero-dimensional closed subscheme of Y. Suppose also that for all pairs of points $s_1, s_2 \in S$,*

$$\mathcal{Q}_{s_1} \cong \mathcal{Q}_{s_2} \quad \Rightarrow \quad s_1 = s_2. \tag{4}$$

Then for some point $s \in S$, the Kodaira–Spencer map for the family $\mathcal{Q}$ at s is injective.

Proof. First, we may suppose that S is affine. Take a point $s \in S$. By the theorem on cohomology and base-change, the natural map

$$H^0(S \times Y, \mathcal{Q}) \longrightarrow H^0(Y, \mathcal{Q}_s)$$

is surjective, so we can find a section $g\colon \mathcal{O}_{S \times Y} \to \mathcal{Q}$ such that the restriction $g_s\colon \mathcal{O}_Y \to \mathcal{Q}_s$ is surjective. Passing to an open subset of S, we can assume that g is surjective, so that $\mathcal{Q}$ is the structure sheaf of a closed subscheme of $S \times Y$.

Let P be the (constant) Hilbert polynomial of the sheaf $\mathcal{Q}_s$ on Y. By the general existence theorem for Hilbert schemes [**6**], there is a scheme $\mathrm{Hilb}^P(Y)$ representing the functor which assigns to a scheme S the set of S-flat quotients $\mathcal{Q}$ of $\mathcal{O}_{S \times Y}$ with Hilbert polynomial P. Let $\mathcal{E}$ be the universal quotient on $\mathrm{Hilb}^P(Y) \times Y$. Then there is a morphism $f\colon S \to \mathrm{Hilb}^P(Y)$ such that $\mathcal{Q} = (f \times 1_Y)^*(\mathcal{E})$. The Kodaira–Spencer map for the family $\mathcal{Q}$ at $s \in S$ is obtained by composing the Kodaira–Spencer map for the family $\mathcal{E}$ at $f(s)$ with the differential

$$\mathrm{T}_s(f)\colon \mathrm{T}_s S \longrightarrow \mathrm{T}_{f(s)} \mathrm{Hilb}^P(Y).$$

Now condition (4) implies that the morphism f is injective on points, and it follows that for general $s \in S$, $\mathrm{T}_s(f)$ is injective. Finally, the fact that the Kodaira–Spencer map for the family $\mathcal{E}$ is injective is a consequence of the universal property of $\mathcal{E}$. This completes the proof.

Theorem 3.3 allows us to say when F is an equivalence.

THEOREM 5.4. *Suppose that F is fully faithful. Then F is an equivalence if and only if for every point $y \in Y$,*

$$F\mathcal{O}_y \otimes \omega_X \cong F\mathcal{O}_y. \tag{5}$$

Proof. Let G and H denote the left and right adjoint functors of F, respectively. Suppose first that F is an equivalence. Then G and H are both quasi-inverses for F, so for any $y \in Y$,

$$G(F\mathcal{O}_y) \cong H(F\mathcal{O}_y) \cong \mathcal{O}_y.$$

From the formulas for G and H given in Lemma 4.5,

$$G(F\mathcal{O}_y) \cong G(F\mathcal{O}_y) \otimes \omega_Y \cong H(F\mathcal{O}_y \otimes \omega_X)[\dim X - \dim Y].$$

But G is an equivalence, so one concludes that X and Y have the same dimension, and there is an isomorphism (5).

For the converse, let X have dimension n, and suppose that (5) holds for all $y \in Y$. Take an object b of $D(X)$ such that $Hb \cong 0$. For any point $y \in Y$, and any integer i,

$$\begin{aligned}\mathrm{Hom}^i_{D(Y)}(Gb, \mathcal{O}_y) &= \mathrm{Hom}^i_{D(X)}(b, F\mathcal{O}_y)\\ &= \mathrm{Hom}^i_{D(X)}(b, F\mathcal{O}_y \otimes \omega_X)\\ &= \mathrm{Hom}^{n-i}_{D(X)}(F\mathcal{O}_y, b)^\vee\\ &= \mathrm{Hom}^{n-i}_{D(Y)}(\mathcal{O}_y, Hb)^\vee = 0,\end{aligned}$$

so, by Example 2.2, $Gb \cong 0$. Applying Theorem 3.3 completes the proof.

Finally, note that Theorems 5.1 and 5.4 imply Theorem 1.1 in the special case when $\mathcal{P}$ is a vector bundle on $X \times Y$.

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