FOURIER-MUKAI TRANSFORMS FOR QUOTIENT VARIETIES

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Abstract. We study Fourier-Mukai transforms for non-singular projective varieties whose canonical bundles have finite order. Our results lead to new transforms for Enriques and bielliptic surfaces.

1. Introduction

Fourier-Mukai transforms are now well-established as a useful tool for computing moduli spaces of sheaves on non-singular projective varieties [2], [9]. Further interest in these transforms has been generated by the appearance of derived categories of coherent sheaves in the homological mirror symmetry conjecture [8]. Despite this, comparatively few Fourier-Mukai transforms are known, and it is therefore of considerable interest to construct new examples.

In this paper we study Fourier-Mukai transforms for complex projective varieties whose canonical bundles have finite order, and relate them to equivariant transforms on certain finite covering spaces. Applying our results to the case of Enriques and bielliptic surfaces, we obtain new examples of transforms for complex surfaces. These results are used in [4], where we find all pairs of minimal complex surfaces with equivalent derived categories.

A Fourier-Mukai (FM) transform is an exact equivalence

$$\Phi : \mathrm{D}(Y) \longrightarrow \mathrm{D}(X)$$

between the bounded derived categories of coherent sheaves on two non-singular projective varieties X and Y. A result of D. Orlov [13] states that for any such equivalence Φ there is an object $\mathcal{P}$ of $\mathrm{D}(Y \times X)$ such that

$$\Phi(-) \cong \mathbf{R}\pi_{X,*}(\mathcal{P} \overset{\mathbf{L}}{\otimes} \pi_Y^*(-)),$$

where $Y \xleftarrow{\pi_Y} Y \times X \xrightarrow{\pi_X} X$ are the projection maps.

Suppose X is a non-singular complex projective variety whose canonical bundle ω_X has finite order n. There is a finite unbranched cover of X by a non-singular projective variety $\widetilde{X}$ with trivial canonical bundle, and X is the quotient of $\widetilde{X}$ by an action of the cyclic group of order n. We shall refer to the quotient morphism $p_X : \widetilde{X} \to X$ as the *canonical cover* of X.

If Y is another non-singular projective variety, and

$$\Phi : \mathrm{D}(Y) \longrightarrow \mathrm{D}(X)$$

is a FM transform, we shall show that ω_Y also has order n, and that if $p_Y : \widetilde{Y} \to Y$ is the canonical cover of Y, then there is a $\mathbb{Z}_n$-equivariant FM transform

$$\widetilde{\Phi} : \mathrm{D}(\widetilde{Y}) \longrightarrow \mathrm{D}(\widetilde{X}),$$

such that the following two squares of functors commute

$$\begin{array}{ccc} \mathrm{D}(\widetilde{Y}) & \xrightarrow{\widetilde{\Phi}} & \mathrm{D}(\widetilde{X}) \\ p_Y^* \uparrow\downarrow p_{Y,*} & & p_X^* \uparrow\downarrow p_{X,*} \\ \mathrm{D}(Y) & \xrightarrow{\Phi} & \mathrm{D}(X). \end{array}$$

Conversely, if $\widetilde{Y}$ is a non-singular projective variety, with trivial canonical bundle, and a free $\mathbb{Z}_n$-action with quotient Y, and

$$\widetilde{\Phi} : \mathrm{D}(\widetilde{Y}) \longrightarrow \mathrm{D}(\widetilde{X})$$

is an equivariant FM transform, we shall show that the quotient map $p_Y : \widetilde{Y} \to Y$ is a canonical cover, and that there is a FM transform Φ such that the diagram above commutes.

Notation. All varieties will be over the complex numbers $\mathbb{C}$. By a sheaf on a variety X we mean a coherent $\mathcal{O}_X$-module, and a point of X always means a closed (or geometric) point. The structure sheaf of such a point $x \in X$ will be denoted $\mathcal{O}_x$.

The bounded derived category of coherent sheaves on a variety X is denoted $\mathrm{D}(X)$. Its objects are complexes of $\mathcal{O}_X$-modules with bounded and coherent cohomology sheaves. The translation (or shift) functor is written [1], so that the symbol $E[m]$ means the object E of $\mathrm{D}(X)$ shifted to the left by m places.

The canonical bundle of a non-singular projective variety X is denoted ω_X. An object E of $\mathrm{D}(X)$ will be called *special* if $E \otimes \omega_X \cong E$ and *simple* if $\mathrm{Hom}_{\mathrm{D}(X)}(E, E) = \mathbb{C}$.

2. Fourier-Mukai transforms

Throughout this section X and Y will be non-singular projective varieties. A Fourier-Mukai (FM) transform relating X and Y is an exact equivalence of categories

$$\Phi : \mathrm{D}(Y) \longrightarrow \mathrm{D}(X).$$

Here exact means commuting with the translation functors and taking triangles to triangles. In this section we list some basic properties of FM transforms. We start with the following simple consequence of Serre duality. For a proof see [4, Lemma 2.1].

Lemma 2.1. *If there is a FM transform $\mathrm{D}(Y) \to \mathrm{D}(X)$ then the canonical bundles of X and Y have the same order.* $\square$

Given an object $\mathcal{P}$ of $\mathrm{D}(Y \times X)$ there is an exact functor

$$\Phi^{\mathcal{P}}_{Y \to X} : \mathrm{D}(Y) \longrightarrow \mathrm{D}(X)$$

defined by the formula

$$\Phi^{\mathcal{P}}_{Y \to X}(-) = \mathbf{R}\pi_{X,*}(\mathcal{P} \overset{\mathbf{L}}{\otimes} \pi_Y^*(-)),$$

as in the introduction. Functors of this type will be called *integral functors.* It is easily checked [10, Proposition 1.3], that the composite of two integral functors is again an integral functor.

It was proved by Orlov [13, Theorem 2.2] that any FM transform is isomorphic to an integral functor. Moreover, the object $\mathcal{P}$ of $\mathrm{D}(Y\times X)$ defining the functor is unique up to isomorphism. We refer to it as the *kernel* of the transform.

Note that $\mathcal{P}$ is always a simple object of $\mathrm{D}(Y\times X)$. This is because there is a relative transform

$$\Phi_Y\colon\ \mathrm{D}(Y\times Y)\longrightarrow \mathrm{D}(Y\times X)$$

which takes the structure sheaf of the diagonal $\Delta\subset Y\times Y$ to the object $\mathcal{P}$. Thus

$$\mathrm{Hom}_{\mathrm{D}(Y\times X)}(\mathcal{P},\mathcal{P})=\mathrm{Hom}_{\mathrm{D}(Y\times Y)}(\mathcal{O}_\Delta,\mathcal{O}_\Delta)=\mathbb{C}.$$

Note also that there is an isomorphism

$$\mathcal{P}\otimes\pi_X^*(\omega_X)\cong\mathcal{P}\otimes\pi_Y^*(\omega_Y) \tag{1}$$

because, up to shifts, these objects are the kernels of the left and right adjoint functors of Φ respectively (see e.g. [3, Lemma 4.5]), which are isomorphic because Φ is an equivalence.

Suppose one has a FM transform $\Phi:\mathrm{D}(Y)\to\mathrm{D}(X)$ such that for each point $y\in Y$ there is a point $f(y)\in X$ with

$$\Phi(\mathcal{O}_y)=\mathcal{O}_{f(y)}.$$

Then f defines a morphism $Y\to X$, and for some line bundle L on Y, there is an isomorphism of functors

$$\Phi(-)\cong f_*(L\otimes -).$$

To see this note that by [3, Lemma 4.3], the kernel $\mathcal{P}$ of Φ is a sheaf on $Y\times X$, flat over Y, such that for each $y\in Y$, $\mathcal{P}_y=\mathcal{O}_{f(y)}$. But if $\Delta\subset X\times X$ denotes the diagonal, then the sheaf $\mathcal{O}_\Delta$ is a universal sheaf parameterising structure sheaves of points of X. It follows that f is a morphism of varieties, and

$$\mathcal{P}=(f\times\mathrm{id}_X)^*(\mathcal{O}_\Delta)\otimes\pi_Y^*(L)$$

for some line bundle L on Y. The claim follows.

Many examples of FM transforms for surfaces are constructed using the following theorem, which is a simple consequence of the results of [3]. See [4, Corollary 2.8] for a proof.

Theorem 2.2. *Let X be a non-singular projective surface with a fixed polarisation and let Y be a two-dimensional, complete, non-singular, fine moduli space of stable, special sheaves on X. Then there is a universal sheaf $\mathcal{P}$ on $Y\times X$ and the resulting functor $\Phi=\Phi^{\mathcal{P}}_{Y\to X}$ is a FM transform.* □

We shall need the following well-known observation. Suppose we are in the situation of Theorem 2.2, and suppose that E is a stable sheaf on X with the same Chern character as the sheaves $\mathcal{P}_y$. We claim that E must be isomorphic to one of the $\mathcal{P}_y$. If not, for each $y\in Y$ we must have

$$\mathrm{Hom}_X(E,\mathcal{P}_y)=\mathrm{Hom}_X(\mathcal{P}_y,E)=0.$$

Since $\mathcal{P}_y$ is special, Serre duality implies that $\mathrm{Ext}^2_X(E,\mathcal{P}_y)=0$, and since E has the same Chern character as $\mathcal{P}_y$, and Φ is an equivalence,

$$\chi(E,\mathcal{P}_y)=\chi(\mathcal{P}_y,\mathcal{P}_y)=\chi(\Phi(\mathcal{O}_y),\Phi(\mathcal{O}_y))=\chi(\mathcal{O}_y,\mathcal{O}_y)=0,$$

so this is enough to show that $\mathrm{Hom}^i_X(E,\mathcal{P}_y)=0$ for all i. This is impossible by [3, Example 2.2], because if Ψ is the inverse of Φ,

$$\mathrm{Hom}^i_X(E,\mathcal{P}_y)=\mathrm{Hom}^i_Y(\Psi(E),\mathcal{O}_y).$$

We conclude this section with a couple of well-known examples of FM transforms which will be useful later.

Example 2.3. The first example of a FM transform for a K3 surface was the reflection functor of [11], although Mukai never explicitly mentions the fact that it is an equivalence of derived categories. To construct it, take a K3 surface X and let $\mathcal{P}$ be the ideal sheaf $\mathcal{I}_\Delta$ of the diagonal in $X\times X$. For any $x\in X$ one has $\mathcal{P}_x\cong\mathcal{I}_x$. Theorem 2.2 shows that the functor $\Phi^{\mathcal{P}}_{X\to X}$ is a FM transform.

Example 2.4. Let (X,ℓ) be a principally polarised abelian surface, and let Y be the moduli space of stable sheaves on X of Chern character $(4,2\ell,1)$. This moduli space is fine, complete and two-dimensional, so there is a universal sheaf $\mathcal{P}$ on $Y\times X$, and the resulting functor $\Phi^{\mathcal{P}}_{Y\to X}$ is a FM transform. In fact Y is isomorphic to X. See [9, Proposition 7.1] for details.

3. Equivariant sheaves and cyclic covers

Let G be a finite group acting freely on a non-singular projective variety $\widetilde{X}$. The quotient variety $X=\widetilde{X}/G$ is non-singular and projective and there is a quotient morphism $p:\widetilde{X}\to X$.

A G-equivariant sheaf (or G-sheaf) on $\widetilde{X}$ is a sheaf $\widetilde{E}$ on $\widetilde{X}$ together with isomorphisms $\lambda_g\colon\widetilde{E}\to g^*(\widetilde{E})$ for each $g\in G$, such that $\lambda_1=\mathrm{id}_{\widetilde{E}}$ and such that for any pair $g,h\in G$,

$$\lambda_{hg}=g^*(\lambda_h)\circ\lambda_g. \tag{2}$$

A morphism of G-sheaves is a morphism $f\colon\widetilde{E}\to\widetilde{F}$ of the underlying sheaves commuting with the given isomorphisms:

$$\lambda^{\widetilde{F}}_g\circ f=g^*(f)\circ\lambda^{\widetilde{E}}_g.$$

It is well-known [5, Chap. V] that the functor p^* defines an equivalence between the category of sheaves on X and the category of G-sheaves on $\widetilde{X}$. What we want to do now is to show that a corresponding result holds for objects of the derived category.

Let us define a G-object of $\mathrm{D}(\widetilde{X})$ to be an object $\widetilde{E}$ together with isomorphisms $\lambda_g\colon\widetilde{E}\to g^*(\widetilde{E})$ in $\mathrm{D}(\widetilde{X})$ for each $g\in G$ satisfying the relations (2).

Proposition 3.1. *For every G-object $\widetilde{E}$ of $\mathrm{D}(\widetilde{X})$ there is an object E of $\mathrm{D}(X)$ such that $\widetilde{E}\cong p^*(E)$.*

Proof. We proceed by induction on the number r of non-zero cohomology sheaves of $\widetilde{E}$. If $r=1$ then up to shifts $\widetilde{E}$ is a G-sheaf on $\widetilde{X}$ as above, so we may assume $r>1$. Shifting $\widetilde{E}$ to the left or right we may as well assume that $H^i(\widetilde{E})=0$ for $i>0$ and that $H^0(\widetilde{E})$ is non-zero.

Let $\tau_{<0}$ and $\tau_{\geq 0}$ be the truncation functors of the standard t-structure on $\mathrm{D}(\widetilde{X})$. There is a triangle

$$\tau_{<0}(\widetilde{E}) \longrightarrow \widetilde{E} \longrightarrow \tau_{\geq 0}(\widetilde{E}) \longrightarrow \tau_{<0}(\widetilde{E})[1],$$

where $\widetilde{A} = \tau_{\geq 0}(\widetilde{E}) = H^0(\widetilde{E})$ is a sheaf. Putting $\lambda_g^{\widetilde{A}} = \tau_{\geq 0}(\lambda_g^{\widetilde{E}})$ makes $\widetilde{A}$ into a G-sheaf, so there is a sheaf A on X with $\widetilde{A} = p^*A$. Similarly $\tau_{<0}(\widetilde{E})$ is a G-object, and since it has $r-1$ non-zero cohomology sheaves, we may apply our induction hypothesis and conclude that it is the pull-back of an object F of $\mathrm{D}(X)$. Thus there is a triangle

$$p^*F \longrightarrow \widetilde{E} \longrightarrow p^*A \xrightarrow{\tilde{f}} p^*F[1].$$

We claim that there is a morphism $f\colon A \to F[1]$ in $\mathrm{D}(X)$ such that $\tilde{f} = p^*(f)$. Assuming this, note that the cone E of f then satisfies $\widetilde{E} \cong p^*E$ and we are done.

To prove the claim replace F by an injective resolution

$$\cdots \longrightarrow I^{-1} \xrightarrow{d^{-1}} I^0 \xrightarrow{d^0} I^1 \longrightarrow \cdots,$$

as in [6, Lemma I.4.6] and consider the map

$$\alpha\colon \operatorname{Hom}_{\widetilde{X}}(p^*A, p^*I^0) \longrightarrow \operatorname{Hom}_{\widetilde{X}}(p^*A, p^*I^1).$$

The group G acts on both of these vector spaces and α is equivariant. Thus each space decomposes as a direct sum over the irreducible representations of G and α respects these decompositions.

The complex $p^*I^\bullet$ is an injective resolution of p^*F so the morphism $\tilde{f}$ is represented by an element $v \in \operatorname{Hom}_{\widetilde{X}}(p^*A, p^*I^1)$. Decomposing v we can write $v = v_0 + v_1$ where v_0 is fixed by the action of G and v_1 is in the span of the other eigenspaces. The fact that $\tilde{f}$ is equivariant means that v_1 lies in the image of α. Thus $\tilde{f}$ is also represented by v_0. But then $\tilde{f}$ is represented by an equivariant element of $\operatorname{Hom}_{\widetilde{X}}(p^*A, p^*I^1)$ and hence is equal to $p^*(u)$ for some $u \in \operatorname{Hom}_X(M, I^1)$. ☐

The group actions we consider in this paper arise from the following standard cyclic cover construction.

Proposition 3.2. *Let X be a non-singular projective variety with a line bundle L of finite order n. Then there is a non-singular projective variety $\widetilde{X}$ and an étale cover $p : \widetilde{X} \to X$ of degree n, such that*

$$p_*(\mathcal{O}_{\widetilde{X}}) = \bigoplus_{i=0}^{n-1} L^{\otimes i}. \tag{3}$$

Furthermore, $\widetilde{X}$ is uniquely defined up to isomorphism, and there is a free action of the cyclic group $G = \mathbb{Z}_n$ on $\widetilde{X}$ such that $X = \widetilde{X}/G$ and $p : \widetilde{X} \to X$ is the quotient morphism.

Proof. By the results of [1, §I.17] there exists a non-singular projective variety $\widetilde{X}$ and a degree n unbranched cover satisfying (3). By [7, Exercise

II.5.17], $\widetilde{X}$ is isomorphic to $\mathbf{Spec}(\mathcal{A})$, where $\mathcal{A}$ is the $\mathcal{O}_X$-algebra

$$\bigoplus_{i=0}^{n-1} L^{\otimes i},$$

which proves uniqueness. The action of G is generated by the automorphism $\otimes L$ of $\mathcal{A}$, and clearly $X = \widetilde{X}/G$. $\square$

Proposition 3.3. *Take assumptions as in Proposition 3.2. Thus X is the quotient of $\widetilde{X}$ by the cyclic group $G = \mathbb{Z}_n$. Let g be a generator of G. Then*

(a) *A simple object $\widetilde{E}$ of $\mathrm{D}(\widetilde{X})$ is isomorphic to $p^*(E)$ for some object E of $\mathrm{D}(X)$ if and only if there is an isomorphism $g^*(\widetilde{E}) \cong \widetilde{E}$.*

(b) *A simple object E of $\mathrm{D}(X)$ is isomorphic to $p_*(\widetilde{E})$ for some object $\widetilde{E}$ of $\mathrm{D}(\widetilde{X})$ if and only if there is an isomorphism $E \otimes L \cong E$.*

Proof. First, part (a). Suppose $\lambda_g \colon \widetilde{E} \to g^*(\widetilde{E})$ is an isomorphism. Since G is cyclic and $\widetilde{E}$ is simple, replacing λ_g by a suitable scalar multiple we can define the structure of a G-object on $\widetilde{E}$. The result then follows from Proposition 3.1.

Part (b) is entirely analogous. The functor p_* effects an equivalence between the category of $\mathcal{O}_{\widetilde{X}}$-modules and the category of modules over the sheaf of algebras $\mathcal{A}$ of Proposition 3.2. A module over this algebra is nothing but an $\mathcal{O}_X$-module E together with compatible isomorphisms

$$E \otimes L^{\otimes i} \to E \otimes L^{\otimes j}$$

for all pairs $i, j \in \mathbb{Z}_n$. Again, if E is simple, it is enough to have an isomorphism $E \otimes L \to E$. The corresponding statement for objects of $\mathrm{D}(X)$ follows as in Proposition 3.1. $\square$

We make the following definition.

Definition 3.4. Let X be a non-singular projective variety whose canonical bundle has finite order n. By the *canonical cover* of X we shall mean the unique non-singular projective variety $\widetilde{X}$ of Proposition 3.2 together with the quotient morphism $p_X : \widetilde{X} \to X$.

The reader should note that the action of $\mathbb{Z}_n$ on the canonical cover $\widetilde{X}$ is only defined up to automorphisms of $\mathbb{Z}_n$.

4. Lifts of FM transforms

In this section we prove our main result, relating FM transforms on varieties with canonical bundles of finite order to equivariant FM transforms on the canonical covers. Throughout we shall suppose that the cyclic group $G = \mathbb{Z}_n$ acts freely on two non-singular projective varieties $\widetilde{X}$ and $\widetilde{Y}$, and denote the quotient morphisms by $p_X : \widetilde{X} \to X$ and $p_Y : \widetilde{Y} \to Y$ respectively.

Definition 4.1. A functor $\widetilde{\Phi} : \mathrm{D}(\widetilde{Y}) \to \mathrm{D}(\widetilde{X})$ is called *equivariant* if there are isomorphisms of functors

$$g^* \circ \widetilde{\Phi} \cong \widetilde{\Phi} \circ g^*$$

for each $g \in G$.

Definition 4.2. Given a functor $\Phi : \mathrm{D}(Y) \to \mathrm{D}(X)$, a *lift* of Φ is a functor $\widetilde{\Phi} : \mathrm{D}(\widetilde{Y}) \to \mathrm{D}(\widetilde{X})$ such that the following two squares of functors commute up to isomorphism

$$\begin{array}{ccc} \mathrm{D}(\widetilde{Y}) & \xrightarrow{\widetilde{\Phi}} & \mathrm{D}(\widetilde{X}) \\ p_Y^* \uparrow \downarrow p_{Y,*} & & p_X^* \uparrow \downarrow p_{X,*} \\ \mathrm{D}(Y) & \xrightarrow{\Phi} & \mathrm{D}(X), \end{array}$$

that is such that there are isomorphisms of functors

$$p_{X,*} \circ \widetilde{\Phi} \cong \Phi \circ p_{Y,*}, \qquad p_X^* \circ \Phi \cong \widetilde{\Phi} \circ p_Y^*. \tag{4}$$

We also say that $\widetilde{\Phi}$ *descends* to give the functor Φ.

We start with an easy lemma.

Lemma 4.3. *Let* $\Phi : \mathrm{D}(X) \to \mathrm{D}(X)$ *and* $\widetilde{\Phi} : \mathrm{D}(\widetilde{X}) \to \mathrm{D}(\widetilde{X})$ *be integral functors such that* $\widetilde{\Phi}$ *lifts* Φ.

(a) *Suppose* $\Phi \cong \mathrm{id}_{\mathrm{D}(X)}$. *Then* $\widetilde{\Phi} \cong g_*$ *for some* $g \in G$. *In particular* $\widetilde{\Phi}$ *is an equivalence.*

(b) *Suppose* $\widetilde{\Phi} \cong \mathrm{id}_{\mathrm{D}(\widetilde{X})}$. *Then* Φ *is an equivalence. If* p_X *is a canonical cover, then* $\Phi(-) \cong (\omega_X^{\otimes i} \otimes -)$ *for some integer* i.

Proof. Write $p = p_X$. We start with (a). Take a point $\tilde{x} \in \widetilde{X}$, and put $x = p(\tilde{x})$. The object $E = \widetilde{\Phi}(\mathcal{O}_{\tilde{x}})$ satisfies $p_*(E) = \mathcal{O}_x$, so $E = \mathcal{O}_{f(\tilde{x})}$ for some point $f(\tilde{x})$ in the fibre $p^{-1}(x)$. As we observed in Section 2, this implies that $f : \widetilde{X} \to \widetilde{X}$ is a morphism of varieties, and for some line bundle L on $\widetilde{X}$,

$$\widetilde{\Phi}(-) \cong f_*(L \otimes -).$$

Since $f(\tilde{x})$ always lies in the fibre $p^{-1}(x)$, one has $f = g$ for some $g \in G$. The functor $g_*^{-1} \circ \widetilde{\Phi}$ also lifts the identity, and takes $p^*(\mathcal{O}_X) = \mathcal{O}_{\widetilde{X}}$ to L, so in fact L must be trivial.

To prove (b), take a point $x \in X$, and a point $\tilde{x} \in \widetilde{X}$ such that $p(\tilde{x}) = x$. Then

$$\Phi(\mathcal{O}_x) = p_*(\mathcal{O}_{\tilde{x}}) = \mathcal{O}_x$$

so that $\Phi \cong (L \otimes -)$ for some line bundle L on X. We must have $p^* L = \mathcal{O}_{\widetilde{X}}$, so if p is the canonical cover, the projection formula gives

$$L \otimes \Big(\bigoplus_{i=0}^{n-1} \omega_X^{\otimes i}\Big) = L \otimes p_*(p^* \mathcal{O}_X) = p_*(p^* L) = \bigoplus_{i=0}^{n-1} \omega_X^{\otimes i},$$

and L is a power of ω_X. $\square$

Consider the skew-diagonal action of G on $\widetilde{Y} \times \widetilde{X}$ given by

$$g(\tilde{y}, \tilde{x}) = (g(\tilde{y}), g^{-1}(\tilde{x})).$$

There is a quotient morphism $f : \widetilde{Y} \times \widetilde{X} \to Z$ and the group G acts freely on Z with quotient $Y \times X$. There is a commutative diagram

$$\begin{tikzcd}
& \widetilde{Y} \times \widetilde{X} \arrow[dl, "\pi_{\widetilde{Y}}"'] \arrow[d, "f"] \arrow[dr, "\pi_{\widetilde{X}}"] & \\
\widetilde{Y} \arrow[d, "p_Y"'] & Z \arrow[dl, "j"'] \arrow[d, "h"] \arrow[dr, "k"] & \widetilde{X} \arrow[d, "p_X"] \\
Y & Y \times X \arrow[l, "\pi_Y"] \arrow[r, "\pi_X"'] & X
\end{tikzcd}$$

Lemma 4.4. *Let $\mathcal{Q}$ be an object of $\mathrm{D}(Z)$ and put $\widetilde{\mathcal{P}} = f^*(\mathcal{Q})$ and $\mathcal{P} = h_*(\mathcal{Q})$. Then $\widetilde{\Phi} = \Phi^{\widetilde{\mathcal{P}}}_{\widetilde{Y}\to\widetilde{X}}$ is a lift of $\Phi = \Phi^{\mathcal{P}}_{Y\to X}$.*

Proof. Let E be an object of $\mathrm{D}(Y)$. By [6, II.5.6, II.5.12], there are natural isomorphisms

$$\begin{aligned}
\widetilde{\Phi}(p_Y^*(\widetilde{E})) &\cong \mathbf{R}\pi_{\widetilde{X},*}(\widetilde{\mathcal{P}} \overset{\mathbf{L}}{\otimes} \pi_{\widetilde{Y}}^* p_Y^*(E)) \cong \mathbf{R}\pi_{\widetilde{X},*}(f^*(\mathcal{Q} \overset{\mathbf{L}}{\otimes} j^*(E))) \\
&\cong p_X^* \, \mathbf{R}k_*(\mathcal{Q} \overset{\mathbf{L}}{\otimes} j^*(E))) \cong p_X^* \, \mathbf{R}\pi_{X,*}(h_*(\mathcal{Q} \overset{\mathbf{L}}{\otimes} h^*\pi_Y^*(E))) \\
&\cong p_X^* \, \mathbf{R}\pi_{X,*}(\mathcal{P} \overset{\mathbf{L}}{\otimes} \pi_Y^*(E)) \cong p_X^* \Phi(E).
\end{aligned}$$

The other isomorphism of (4) can be proved in the same way, or by taking adjoints. $\square$

The following theorem is the main result of this paper.

Theorem 4.5. *Let X and Y be non-singular projective varieties with canonical bundles of order n, and take canonical covers*

$$p_X : \widetilde{X} \to X, \qquad p_Y : \widetilde{Y} \to Y.$$

Thus X and Y are quotients of $\widetilde{X}$ and $\widetilde{Y}$ by the cyclic group $G = \mathbb{Z}_n$. Then any equivariant FM transform

$$\widetilde{\Phi} : \mathrm{D}(\widetilde{Y}) \longrightarrow \mathrm{D}(\widetilde{X})$$

is the lift of some FM transform

$$\Phi : \mathrm{D}(Y) \longrightarrow \mathrm{D}(X),$$

and conversely, we may choose the actions of G on $\widetilde{Y}$ and $\widetilde{X}$ so that any FM transform Φ lifts to give an equivariant FM transform $\widetilde{\Phi}$.

Proof. Let $\widetilde{\Phi} : \mathrm{D}(\widetilde{Y}) \to \mathrm{D}(\widetilde{X})$ be an equivariant transform with kernel $\widetilde{\mathcal{P}}$. The equivariance is equivalent to the existence of isomorphisms

$$(\mathrm{id}_{\widetilde{Y}} \times g)^*(\widetilde{\mathcal{P}}) \cong (g \times \mathrm{id}_{\widetilde{X}})^*(\widetilde{\mathcal{P}})$$

for all $g \in G$. The argument of Proposition 3.3 implies that $\widetilde{\mathcal{P}} = f^*\mathcal{Q}$ for some object $\mathcal{Q}$ of $\mathrm{D}(Z)$. Thus $\widetilde{\Phi}$ descends to a functor Φ.

We must show that Φ is an equivalence. Let $\widetilde{\Psi}$ be a inverse of $\widetilde{\Phi}$. Then $\widetilde{\Psi}$ is equivariant and hence is the lift of some integral functor $\Psi : \mathrm{D}(X) \to \mathrm{D}(X)$. But then the composite functor $\widetilde{\Psi} \circ \widetilde{\Phi} \cong \mathrm{id}_{\mathrm{D}(\widetilde{Y})}$ lifts $\Psi \circ \Phi$, so by Lemma

4.3, twisting Ψ by some power of ω_X, one has $\Psi \circ \Phi \cong \mathrm{id}_{\mathrm{D}(Y)}$. Similarly $\Phi \circ \Psi \cong \mathrm{id}_{\mathrm{D}(X)}$, so Φ is an equivalence.

For the converse, we may take the actions of G on $\widetilde{Y}$ and $\widetilde{X}$ to be such that

$$h_*(\mathcal{O}_Z) = \bigoplus_{i=0}^{n-1} (\pi_X^* \omega_X \otimes \pi_Y^* \omega_Y^{-1})^{\otimes i}.$$

Let $\Phi : \mathrm{D}(Y) \to \mathrm{D}(X)$ be a FM transform with kernel $\mathcal{P}$. It follows from the isomorphism (1) and Proposition 3.3 that $\mathcal{P} = h_* \mathcal{Q}$ for some object $\mathcal{Q}$ of $\mathrm{D}(Z)$. By Lemma 4.4, Φ has a lift $\widetilde{\Phi}$, and furthermore $\widetilde{\Phi}$ is equivariant with respect to our chosen G-actions.

Let Ψ be the inverse of Φ. Then Ψ is also a FM transform and hence lifts to a functor $\widetilde{\Psi} : \mathrm{D}(\widetilde{Y}) \to \mathrm{D}(\widetilde{X})$ by the same argument. The functor $\widetilde{\Psi} \circ \widetilde{\Phi}$ is a lift of $\Psi \circ \Phi \cong \mathrm{id}_{\mathrm{D}(Y)}$, hence, by Lemma 4.3, composing $\widetilde{\Psi}$ with g^* for some $g \in G$, we can assume that $\widetilde{\Psi} \circ \widetilde{\Phi} \cong \mathrm{id}_{\mathrm{D}(\widetilde{Y})}$. Similarly, $\widetilde{\Phi} \circ \widetilde{\Psi} \cong \mathrm{id}_{\mathrm{D}(\widetilde{X})}$ and we are done. □

It is easy to see using Lemma 4.3 that, in the situation of the theorem, if two FM transforms $\widetilde{\Phi}_1, \widetilde{\Phi}_2$ lift a given transform Φ, then $\widetilde{\Phi}_2 \cong g^* \circ \widetilde{\Phi}_1$ for some $g \in G$. Similarly, if FM transforms Φ_1, Φ_2 both lift to give the same transform $\widetilde{\Phi}$, then $\Phi_2 \cong \omega_X^{\otimes i} \otimes \Phi_1$ for some integer i.

A couple of points remain. Let X be a non-singular projective variety whose canonical bundle has order n, and let $p_X : \widetilde{X} \to X$ be the canonical cover. Thus X is the quotient of $\widetilde{X}$ by a free action of $G = \mathbb{Z}_n$.

Firstly, suppose there is a FM transform Φ relating X to another variety Y. Then, by Lemma 2.1, ω_Y also has order n, and taking canonical covers of X and Y we are in the situation of Theorem 4.5.

Secondly, suppose there is a non-singular projective variety $\widetilde{Y}$ with a free G-action, and that there is an equivariant FM transform $\widetilde{\Phi}$ relating $\widetilde{X}$ and $\widetilde{Y}$. Then we claim that the quotient morphism $p_Y : \widetilde{Y} \to Y$ is a canonical cover of $Y = \widetilde{Y}/G$, so we are again in the situation of Theorem 4.5.

To prove the claim, note that by the argument used in the proof of Theorem 4.5, the functor $\widetilde{\Phi}$ descends to give a FM transform $\Phi : \mathrm{D}(Y) \to \mathrm{D}(X)$. Lemma 2.1 shows that ω_Y has order n. Taking a canonical cover Y' of Y we can lift Φ to a FM transform $\Phi' : \mathrm{D}(Y') \to \mathrm{D}(\widetilde{X})$. Now $\widetilde{\Phi}^{-1} \circ \Phi'$ is an equivariant FM transform relating Y' and $\widetilde{Y}$ which lifts the identity on $\mathrm{D}(Y)$. It follows that Y' and $\widetilde{Y}$ are isomorphic as G-spaces, hence the claim.

5. Examples

Let $\widetilde{X}$ be a non-singular projective surface with a fixed polarisation and let $\widetilde{Y}$ be a complete, fine, non-singular, two-dimensional moduli space of stable sheaves on $\widetilde{X}$. Then there is a universal sheaf $\widetilde{\mathcal{P}}$ on $\widetilde{Y} \times \widetilde{X}$, and by Theorem 2.2, the resulting functor

$$\widetilde{\Phi} = \Phi^{\widetilde{\mathcal{P}}}_{\widetilde{Y} \to \widetilde{X}} : \mathrm{D}(\widetilde{Y}) \longrightarrow \mathrm{D}(\widetilde{X})$$

is a FM transform.

Assume that $\widetilde{X}$ has trivial canonical bundle (so is of either abelian or K3 type). By Lemma 2.1 the variety $\widetilde{Y}$ also has trivial canonical bundle.

Suppose further that the cyclic group $G = \mathbb{Z}_n$ acts freely on $\widetilde{X}$ and let $p_X : \widetilde{X} \to X$ denote the quotient morphism.

The action of G on $\widetilde{X}$ induces an action of G on the moduli space $\widetilde{Y}$ such that for each point $\tilde{y} \in \widetilde{Y}$, and each $g \in G$,

$$g^*(\widetilde{\mathcal{P}}_{\tilde{y}}) \cong \widetilde{\mathcal{P}}_{g(\tilde{y})}. \tag{5}$$

If this action of G on $\widetilde{Y}$ is free then we can form the quotient $Y = \widetilde{Y}/G$, and Theorem 4.5 shows that $\widetilde{\Phi}$ descends to give a FM transform $\Phi : \mathrm{D}(Y) \to \mathrm{D}(X)$. The following lemma gives a purely numerical criterion for when this happens.

Lemma 5.1. *The action of G on $\widetilde{Y}$ defined above is free if and only if the integers $\chi(p_X^*(F), \widetilde{\mathcal{P}}_{\tilde{y}})$, as F varies through all locally-free sheaves on X, have no common factor.*

Proof. Assume first that the action of G on $\widetilde{Y}$ is free. As we noted above $\widetilde{\Phi}$ descends to a transform $\Phi : \mathrm{D}(Y) \to \mathrm{D}(X)$. Let Ψ be the inverse of Φ, take a point $\tilde{y} \in \widetilde{Y}$ and put $\widetilde{E} = \widetilde{\mathcal{P}}_{\tilde{y}}$. By the adjunction $p_X^* \dashv p_{X,*}$

$$\chi(p_X^* F, \widetilde{E}) = \chi(F, p_{X,*}(\widetilde{E})).$$

If $y = p_Y(\tilde{y})$, $\Phi(\mathcal{O}_y) = p_{X,*}(\widetilde{E})$, so for any locally-free sheaf F on X

$$\chi(F, p_{X,*}(\widetilde{E})) = \chi(\Psi(F), \mathcal{O}_y).$$

Since Ψ is an equivalence these integers have no common factor.

For the converse suppose that the G-action is not free, so that for some $\tilde{y} \in \widetilde{Y}$, the stabilizer of $\widetilde{E} = \widetilde{\mathcal{P}}_{\tilde{y}}$ is some subgroup $H \subset G$ of order $m > 1$. Then $\widetilde{E} = f^*(A)$ for some A, where f is the intermediate quotient map $f\colon \widetilde{X} \to \widetilde{X}/H$. For any bundle F on $\widetilde{X}/H$

$$\chi(f^*(F), \widetilde{E}) = \chi(f^*(F), f^*(A)) = m\,\chi(F, A)$$

and the result follows. □

Example 5.2. Let X be an Enriques surface. There is a K3 surface $\widetilde{X}$ with an involution σ such that X is the quotient of $\widetilde{X}$ by the group $\mathbb{Z}_2$ generated by σ. For any point $\tilde{x} \in \widetilde{X}$ one has

$$\sigma^*(\mathcal{I}_{\tilde{x}}) = \mathcal{I}_{\sigma(\tilde{x})},$$

so the reflection functor of Example 2.3 descends to give a FM transform $\Phi : \mathrm{D}(X) \to \mathrm{D}(X)$. This has the property that for each $x \in X$ there is an exact sequence

$$0 \longrightarrow \Phi(\mathcal{O}_x) \longrightarrow \mathcal{O}_X \oplus \omega_X \longrightarrow \mathcal{O}_x \longrightarrow 0.$$

It is this transform which was studied by S. Zube [14, §3.7].

Example 5.3. Let X be a bielliptic surface which is a quotient of a product of elliptic curves $\widetilde{X} = C_1 \times C_2$ by a cyclic group $\mathbb{Z}_n$. The quotient map $p\colon \widetilde{X} \to X$ is then the canonical cover of X.

The original Fourier-Mukai functor of [10] never descends because the sheaf $\mathcal{O}_{\widetilde{X}} = \mathcal{F}(\mathcal{O}_0)$ is G-invariant.

Consider instead the moduli space $\widetilde{Y}$ of stable sheaves on $\widetilde{X}$ of Chern character $(4, 2\ell, 1)$, where $\ell = C_1 + C_2$ is a principal polarisation. By Lemma 5.1, the FM transform of Example 2.4 descends to give a FM transform $\Phi : \mathrm{D}(Y) \to \mathrm{D}(X)$, such that for all $y \in Y$ the object $\Phi(\mathcal{O}_y)$ is a locally-free sheaf on X of rank $4n$.

Remark 5.4. In [4] we show that if X and Y are Enriques or bielliptic surfaces, and $\Phi : \mathrm{D}(Y) \to \mathrm{D}(X)$ is a FM transform, then X and Y are isomorphic.

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