

STOKES FACTORS AND MULTILOGARITHMS

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ABSTRACT. Let G be a complex, affine algebraic group and ∇ a meromorphic connection on the trivial G -bundle over \mathbb{P}^1 with a pole of order 2 at zero and a pole of order 1 at infinity. We show that the map \mathcal{S} taking the residue of ∇ at zero to the corresponding Stokes factors is given by an explicit, universal Lie series whose coefficients are multilogarithms. We prove moreover that the same holds for the Taylor series of the inverse of \mathcal{S} and that the corresponding Lie series coincides with the generating function for counting invariants in abelian categories constructed by D. Joyce.

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1. INTRODUCTION

1.1. Let \mathcal{V} be the trivial, rank n complex vector bundle on \mathbb{P}^1 and consider a meromorphic connection on \mathcal{V} of the form

$$\nabla = d - \left(\frac{Z}{t^2} + \frac{f}{t} \right) dt \quad (1)$$

where Z, f are $n \times n$ matrices and Z is diagonal with distinct eigenvalues z_1, \dots, z_n .

Since ∇ has an irregular singularity at $t = 0$, its gauge equivalence class, as a connection on the unit disk, is determined by its *Stokes factors* [1]. These encode the change in the asymptotics of the canonical fundamental solutions of ∇ across

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suitable rays in the t -plane. For each such *Stokes ray*, that is a ray of the form $\ell = \mathbb{R}_{>0}(z_i - z_j)$, the Stokes factor S_ℓ is a unipotent matrix whose only non-zero, off-diagonal entries are of the form $(S_\ell)_{jk}$ with j, k such that $z_j - z_k \in \ell$.

1.2. The computation of the Stokes factors of ∇ was carried out by Balser–Jurkat–Lutz in [2] by using the Fourier–Laplace transform $\widehat{\nabla}$ of ∇ . $\widehat{\nabla}$ is the Fuchsian connection on \mathcal{V} with poles at the points z_1, \dots, z_n , given by

$$\widehat{\nabla} = d - \sum_{i=1}^n \frac{P_i f}{z - z_i} dz \quad (2)$$

where P_i is the projection onto the Z -eigenspace corresponding to z_i .

The canonical fundamental solutions of ∇ are first expressed as linear combinations of Fourier–Laplace transforms of suitable vector solutions of $\widehat{\nabla}$ and the corresponding Stokes factors are then computed in terms of a connection matrix. The entries of this matrix encode the analytic continuation of the latter solutions from one pole to another and are therefore expressible in terms of multilogarithms, although this latter fact is only implicit in [2].

1.3. The first aim of this paper is to extend the results of [1, 2] in two distinct directions by allowing:

- (i) the structure group $GL_n(\mathbb{C})$ of the connection ∇ to be an arbitrary complex, affine algebraic group G .
- (ii) The element Z to be an arbitrary semisimple element of the Lie algebra of G , in particular a diagonal matrix with repeated eigenvalues for $G = GL_n(\mathbb{C})$.

With regard to (i), an extension of the Balser–Jurkat–Lutz theory of invariants for meromorphic connections [1], in particular the construction of canonical fundamental solutions and definition of Stokes data, was carried out by P. Boalch for an arbitrary complex, reductive group [6, 7]. Just as [1], Boalch’s treatment applies to a more general context than the one presented above: it is local, in that it encompasses meromorphic connections on the unit disk rather than \mathbb{P}^1 , and allows for arbitrary order poles. It does not however address the extension (ii).

It turns out that an extension of [1] to the larger class of affine algebraic groups may be readily obtained from the corresponding results for the group $GL_n(\mathbb{C})$ by using Tannaka duality, that is the reconstruction of such groups from their finite-dimensional representations. This requires taking care of the extension (ii) for $G = GL_n(\mathbb{C})$ however, since the assumption that Z be regular semisimple is not stable under passage to a representation.

When $G = GL_n(\mathbb{C})$ and Z is a matrix with repeated eigenvalues, the extension of [1] and [2] was in fact carried out for arbitrary order poles in [3] and [4] respectively¹. The main result of [4], namely the computation of the Stokes factors in terms of the analytic continuation of suitable associated functions is not sufficiently explicit for our purposes however. We therefore give a streamlined treatment which closely parallels that of [2] and relies on the special form of the connection ∇ .

¹We are grateful to Phil Boalch for pointing us to [4].

1.4. Once the canonical fundamental solutions of ∇ are constructed, one finds, as in [1, 6], that each Stokes factor S_ℓ is a unipotent element of G of the form $S_\ell = \exp(\epsilon_\ell)$, where ϵ_ℓ lies in the span of the $\text{ad}(Z)$ -eigenspaces corresponding to eigenvalues lying on ℓ . As ℓ varies through the Stokes rays, the logarithms of the Stokes factors may be assembled to give a *Stokes map* \mathcal{S} mapping f to $\epsilon = \sum_\ell \epsilon_\ell$.

Extending the results of [2], we compute this map explicitly, first for $G = GL_n(\mathbb{C})$ by using the Fourier–Laplace transform and then for an arbitrary algebraic group by using Tannaka duality. Our results differ in their form from those of [2] in that by considering the logarithms of the Stokes factors rather than the Stokes factors themselves, we express the answer as a *universal Lie series* akin to the Baker–Campbell–Hausdorff series. The coefficients of this series are multilogarithms evaluated at the eigenvalues of $\text{ad}(Z)$.

1.5. The Lie series computing \mathcal{S} may be inverted to yield the Taylor series of the local inverse of the Stokes map at $\epsilon = 0$. This inverse is again expressed as an explicit, universal Lie series involving multilogarithms. To the best of our knowledge, this result is new even when $G = GL_n(\mathbb{C})$ and Z has distinct eigenvalues.

The inversion of \mathcal{S} is obtained by using a non-commutative analogue of the compositional inversion of a formal power series. The latter expresses the answer as a sum over plane rooted trees and may be of independent interest.

Remarkably, the Taylor series of \mathcal{S}^{-1} coincides with the generating series for counting invariants in an abelian category \mathcal{A} constructed by D. Joyce. This allows us to reinterpret Joyce’s construction as the statement that a stability condition on \mathcal{A} defines Stokes data for a connection of the form (1) with values in the Ringel–Hall Lie algebra of \mathcal{A} [8].

1.6. We conclude with a more detailed description of the contents of this paper. In Section 2, we review the definition of the canonical fundamental solutions of an irregular connection of the form (1) and of the corresponding Stokes data. In Section 3 we discuss isomonodromic deformations of such connections. In Section 4, we state our results concerning the computation of the Stokes map and of the Taylor series of its inverse in terms of multilogarithms. Section 5 contains similar results for the Stokes multipliers of the connection. The rest of the paper contains the proofs of the results of Section 4. Specifically, in Section 6 we prove the uniqueness of canonical fundamental solutions of the connection ∇ . Section 7 covers mostly well-known background material on the computation of regularised parallel transport for Fuchsian connections on \mathbb{P}^1 in terms of iterated integrals. In Section 8, we prove the existence of the canonical fundamental solutions for the group $GL_n(\mathbb{C})$ and then compute, in Section 9 the corresponding Stokes factors in terms of multilogarithms. As in [2], these results are derived from those of Section 7 by the use of the Fourier–Laplace transform. In Section 10, we prove our main results for an arbitrary affine algebraic group by using Tannaka duality. Section 11 gives the non-commutative generalisation of the compositional inversion of formal power series which is required to invert the Stokes map.

2. IRREGULAR CONNECTIONS AND STOKES PHENOMENA

We review in this section the definition of Stokes data for irregular connections on \mathbb{P}^1 of the form (1). Our exposition follows [6, 7], where we learnt much of this material. As pointed out in the Introduction however, we depart from [6, 7] and the earlier treatment [1] in that we consider connections having as structure group an arbitrary complex algebraic group rather than a complex reductive group.

2.1. Recollections on algebraic groups. We summarise in this paragraph some standard terminology and facts about algebraic groups and refer the reader to [12] for more details.

By an *algebraic group*, we shall always mean an affine algebraic group G over \mathbb{C} . By a *finite-dimensional representation* of G , we shall mean a rational representation, that is a morphism $G \rightarrow GL(V)$, where V is a finite-dimensional complex vector space. An algebraic group always possesses a faithful finite-dimensional representation and may therefore be regarded as a *linear algebraic group*, that is a (Zariski) closed subgroup of some $GL(V)$.

An element $g \in G$ is *semisimple* (resp. *unipotent*) if it acts by a semisimple (resp. unipotent) endomorphism on any finite-dimensional representation of G . Equivalently, g is semisimple (resp. unipotent) if, after embedding G as a closed subgroup of some $GL(V)$, g is a semisimple (resp. unipotent) endomorphism of V . If G is semisimple, then g is semisimple (resp. unipotent) if, and only if, $\text{Ad}(g)$ is a semisimple (resp. unipotent) endomorphism of the Lie algebra \mathfrak{g} of G .

Similarly, an element $Z \in \mathfrak{g}$ is semisimple (resp. nilpotent) if Z acts as a semisimple (resp. nilpotent) endomorphism on any finite-dimensional representation of G or, equivalently, on a faithful representation of G or, when G is semisimple, on the adjoint representation of G .

2.2. The irregular connection ∇ . Let P be the holomorphically trivial, principal G -bundle on \mathbb{P}^1 . We shall be concerned with meromorphic connections on P of the form

$$\nabla = d - \left(\frac{Z}{t^2} + \frac{f}{t} \right) dt. \quad (3)$$

where $Z, f \in \mathfrak{g}$.

Throughout this paper, we assume that the elements Z, f satisfy the following:

(Z) Z is semisimple. In particular, \mathfrak{g} splits as the direct sum

$$\mathfrak{g} = \mathfrak{g}^Z \oplus [Z, \mathfrak{g}] \quad (4)$$

where \mathfrak{g}^Z is the centraliser of Z and $[Z, \mathfrak{g}]$ is the span of the non-zero eigenspaces of $\text{ad}(Z)$.

(f) The projection of f onto \mathfrak{g}^Z corresponding to the decomposition (4) is zero.

If $G = GL_n(\mathbb{C})$ and Z is a diagonal matrix with distinct eigenvalues, condition (f) is the requirement that the diagonal entries of the matrix f be zero.

We will denote the $\text{ad}(Z)$ -eigenspace corresponding to the eigenvalue $\zeta \in \mathbb{C}$ by $\mathfrak{g}_\zeta \subset \mathfrak{g}$ and the subspace $[Z, \mathfrak{g}] \subset \mathfrak{g}$ of ‘off-diagonal’ elements by \mathfrak{g}_{od} . Thus, $\mathfrak{g}^Z = \mathfrak{g}_0$

and

$$\mathfrak{g}_{\text{od}} = \sum_{\zeta \in \mathbb{C}^*} \mathfrak{g}_{\zeta} \tag{5}$$

2.3. The assumptions above differ from those found in the literature in the following ways:

- (i) with regard to (Z) , the assumption usually made is that Z is a *regular* semisimple element that is, if $G = GL_n(\mathbb{C})$, that Z is diagonalisable with distinct eigenvalues. As pointed out in the Introduction, this latter assumption is not stable under passage to a representation and therefore ill-suited to the Tannakian methods employed in Section 10.
- (ii) On the other hand, the assumption (f) is unduly restrictive. A more natural assumption would be to consider the projection f_0 of f onto \mathfrak{g}^Z and to require that the eigenvalues of $\text{ad}(f_0)$ on \mathfrak{g}^Z are not positive integers. We impose the condition (f) however since it simplifies the form of the Stokes data and holds in the context of stability conditions considered in [8].

2.4. **Stokes rays and sectors.**

Definition. A *ray* is a subset of \mathbb{C}^* of the form $\mathbb{R}_{>0} \exp(i\pi\phi)$. The *Stokes rays* of the connection ∇ are the rays $\mathbb{R}_{>0}\zeta$, where ζ ranges over the non-zero eigenvalues of $\text{ad}(Z)$. The *Stokes sectors* are the open regions of \mathbb{C}^* bounded by them. A ray is called *admissible* if it is not a Stokes ray.

Remark. If G is reductive, the eigenvalues of $\text{ad}(Z)$ are invariant under multiplication by -1 . The Stokes rays therefore come in pairs and the Stokes sectors are convex in this case. This need not be the case for an arbitrary algebraic group.

2.5. **Canonical fundamental solutions.** The Stokes data of the connection ∇ are defined using fundamental solutions with prescribed asymptotics. We first recall how these are characterised.

Given a ray r in \mathbb{C} , we denote by \mathbb{H}_r the corresponding half-plane

$$\mathbb{H}_r = \{z = uv : u \in r, \text{Re}(v) > 0\} \subset \mathbb{C}. \tag{6}$$

The following basic result is well-known for $G = GL_n(\mathbb{C})$ and Z regular (see, e.g. [20, pp. 58–61]) and was extended in [7] to the case of complex reductive groups.² It will be proved in Section 10.

Theorem. *Given an admissible ray r , there is a unique holomorphic function $Y_r : \mathbb{H}_r \rightarrow G$ such that*

$$\frac{dY_r}{dt} = \left(\frac{Z}{t^2} + \frac{f}{t} \right) Y_r \tag{7}$$

$$Y_r \cdot e^{Z/t} \rightarrow 1 \quad \text{as } t \rightarrow 0 \text{ in } \mathbb{H}_r \tag{8}$$

Remark. The function $Y_r \cdot e^{Z/t}$ possesses in fact an asymptotic expansion in \mathbb{H}_r with constant term the identity, but we shall not need this stronger property.

²The references [20] and [7] cover however the more general case when ∇ is only defined on a disk around $t = 0$ and has an arbitrary order pole at $t = 0$.

2.6. The uniqueness statement of Theorem 2.5 and the definition of the Stokes data rely upon the following result which will be proved in Section 6 (see [7, Lemma 22] for the case of G reductive and Z regular).

Proposition. *Let r, r' be two rays such that $r \neq -r'$, and $g \in G$ an element such that*

$$e^{-Z/t} \cdot g \cdot e^{Z/t} \rightarrow 1 \text{ as } t \rightarrow 0 \text{ in } \mathbb{H}_r \cap \mathbb{H}_{r'}.$$

Then, g is unipotent and $X = \log(g)$ lies in

$$\bigoplus_{\zeta \in \overline{\Sigma}(r, r')} \mathfrak{g}_\zeta \subset \mathfrak{g},$$

where $\overline{\Sigma}(r, r') \subset \mathbb{C}^$ is the closed convex sector bounded by r and r' .*

Proposition 2.6 implies in particular that if the rays r, r' are admissible and such that the sector $\overline{\Sigma}(r, r')$ does not contain any Stokes rays of ∇ , the element $g \in G$ determined by

$$Y_r(t) = Y_{r'}(t) \cdot g \quad \text{for } t \in \mathbb{H}_r \cap \mathbb{H}_{r'}$$

is equal to 1. It follows in particular that, given a Stokes sector Σ and a convex subsector $\Sigma' \subset \Sigma$, the solutions Y_r , as r varies in Σ' , patch to a fundamental solution $Y_{\Sigma'}$ of (7) possessing the asymptotic property (8) in the *supersector*

$$\widehat{\Sigma}' = \{uv : u \in \Sigma', \operatorname{Re}(v) > 0\} = \bigcup_{r \subset \Sigma'} \mathbb{H}_r.$$

2.7. Stokes factors. Assume now that ℓ is a Stokes ray. Let r_\pm be small clockwise (resp. anticlockwise) perturbations of ℓ such that the convex sector $\overline{\Sigma}(r_-, r_+)$ does not contain any Stokes rays of ∇ other than ℓ .

Definition. The *Stokes factor* S_ℓ corresponding to ℓ is the element of G defined by

$$Y_{r_+}(t) = Y_{r_-}(t) \cdot S_\ell \quad \text{for } t \in \mathbb{H}_{r_+} \cap \mathbb{H}_{r_-}.$$

By Proposition 2.6, the definition of S_ℓ is independent of the choice of r_\pm . Moreover, S_ℓ is unipotent and $\log(S_\ell) \in \bigoplus_{\zeta \in \ell} \mathfrak{g}_\zeta$.

2.8. Stokes multipliers. An alternative but closely related system of invariants are the Stokes multipliers of the connection ∇ . These depend upon the choice of a ray r such that both r and $-r$ are admissible.

Definition. The *Stokes multipliers* of ∇ corresponding to r are the elements $S_\pm \in G$ defined by

$$Y_{r,\pm}(t) = Y_{-r}(t) \cdot S_\pm, \quad t \in \mathbb{H}_{-r}$$

where $Y_{r,+}$ and $Y_{r,-}$ are the analytic continuations of Y_r to \mathbb{H}_{-r} in the anticlockwise and clockwise directions respectively.

By Proposition 2.6, the multipliers S_\pm remain constant under a perturbation of r so long as r and $-r$ do not cross any Stokes rays.

2.9. To relate the Stokes factors and multipliers, set $r = \mathbb{R}_{>0} \exp(i\pi\theta)$ and label the Stokes rays as $\ell_j = \mathbb{R}_{>0} \exp(i\pi\phi_j)$, with $j = 1, \dots, m_1 + m_2$ where

$$\theta < \phi_1 < \dots < \phi_{m_1} < \theta + 1 < \phi_{m_1+1} < \dots < \phi_{m_1+m_2} < \theta + 2$$

The following result is immediate upon drawing a picture

Lemma. *The following holds*

$$S_+ = S_{\ell_{m_1}} \cdots S_{\ell_1} \quad \text{and} \quad S_- = S_{\ell_{m_1+1}}^{-1} \cdots S_{\ell_{m_1+m_2}}^{-1}$$

The Stokes factors therefore determine the Stokes multipliers for any ray r . Conversely, the Stokes multipliers for a single ray r determine all the Stokes factors. This may be proved along the lines of [1, Lemma 2], [7] by noticing that S_{\pm} lie in the unipotent subgroups $N_{\pm} \subset G$ with Lie algebras $\bigoplus_{\zeta \in \mathbb{H}_{\pm ir}} \mathfrak{g}_{\zeta}$ and that these groups may uniquely be written as products of the unipotent subgroups $\exp(\bigoplus_{\zeta \in \ell} \mathfrak{g}_{\zeta})$ corresponding to the Stokes rays ℓ in $\mathbb{H}_{\pm ir}$. We shall instead give explicit formulae expressing the Stokes factors in terms of S_{\pm} in Proposition 5.2 below.

2.10. **Choice of a torus.** It will be convenient in the sequel to choose a torus $H \subset G$ whose Lie algebra \mathfrak{h} contains Z . Let $\mathbb{X}(H) = \text{Hom}_{\mathbb{Z}}(H, \mathbb{C}^*)$ be the group of characters of H and $\mathbb{X}(H) \cong \Lambda \subset \mathfrak{h}^*$ the lattice spanned by the differentials of elements in $\mathbb{X}(H)$. For any $\lambda \in \Lambda$, we denote the unique element of $\mathbb{X}(H)$ with differential λ by e^{λ} . Decompose \mathfrak{g} as

$$\mathfrak{g} = \mathfrak{g}^{\mathfrak{h}} \oplus [\mathfrak{h}, \mathfrak{g}] = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_{\alpha} \quad (9)$$

where $\Phi = \Phi(G, H) \subset \Lambda \setminus \{0\}$ is a finite set and H acts on \mathfrak{g}_{α} via the character e^{α} so that, in particular $\mathfrak{h} \subset \mathfrak{g}_0$. We refer to the elements of Φ as the *roots of G* relative to H . We note that if H is a maximal torus, the set of roots $\Phi(G, H)$ is independent of the choice of H , but we shall not need to assume this.

Set

$$\Phi^Z = \{\alpha \in \Phi \mid Z(\alpha) \neq 0\} \quad (10)$$

so that

$$[Z, \mathfrak{g}] = \bigoplus_{\alpha \in \Phi^Z} \mathfrak{g}_{\alpha} \quad \text{and} \quad \mathfrak{g}^Z = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Phi \setminus \Phi^Z} \mathfrak{g}_{\alpha} \quad (11)$$

Remark. If $f = \sum_{\zeta \in \mathbb{C}} f_{\zeta}$ is the eigenvector decomposition of $f \in \mathfrak{g}$ with respect to $\text{ad}(Z)$, then

$$f_{\zeta} = \sum_{\alpha: Z(\alpha)=\zeta} f_{\alpha} \quad (12)$$

3. ISOMONODROMIC DEFORMATIONS

We discuss in this section isomonodromic deformations of the connection ∇ , that is families of connections of the form (1) where Z and f vary in such a way that the Stokes data remain constant.

3.1. Variations of Z . We wish to vary Z among semisimple elements of \mathfrak{g} in such a way that the decomposition (4) remains constant.

This is readily seen to be the case if Z varies among the regular elements of the Lie algebra \mathfrak{h} of a torus $H \subset G$. Indeed, let $\Phi \subset \mathfrak{h}^*$ be the set of roots of H and set

$$\mathfrak{h}_{\text{reg}} = \mathfrak{h} \setminus \bigcup_{\alpha \in \Phi} \text{Ker}(\alpha)$$

If $Z \in \mathfrak{h}_{\text{reg}}$, Φ^Z is equal to $\Phi(G; H)$ so that the decomposition (4) remains constant as Z varies in $\mathfrak{h}_{\text{reg}}$ by (11).

Conversely, the following holds

Proposition. *Let $Z \in \mathfrak{g}$ be a fixed semisimple element.*

- (i) *The set of semisimple elements $Z' \in \mathfrak{g}$ which give rise to the same decomposition (4) as Z is the set of regular elements in the Lie algebra \mathfrak{h} of a torus $H \subset G$ such that $Z \in \mathfrak{h}$.*
- (ii) *H is the unique torus in G which is maximal for the property that $Z \in \mathfrak{h}_{\text{reg}}$.*

PROOF. (i) Let $\mathfrak{g}^Z \subseteq \mathfrak{g}$ be the centraliser of Z , $\mathcal{Z}(\mathfrak{g}^Z)$ its centre and $\mathfrak{h} \subseteq \mathcal{Z}(\mathfrak{g}^Z)$ the subspace of its semisimple elements. \mathfrak{h} is the Lie algebra of a torus $H \subset G$ obtained as follows. Let $G^Z \subseteq G$ be the centraliser of Z , $G_1^Z \subseteq G$ its identity component and $\mathcal{Z}(G_1^Z)_1$ the identity component of the centre of G_1^Z . This is a connected, commutative algebraic group. Its subgroup H of semisimple elements is therefore a torus with Lie algebra \mathfrak{h} .

Note next that $\mathfrak{g}^Z = \mathfrak{g}^{\mathfrak{h}}$ and $[Z, \mathfrak{g}] = [\mathfrak{h}, \mathfrak{g}]$ so that $Z \in \mathfrak{h}_{\text{reg}}$. Indeed, since $Z \in \mathfrak{h}$, we have $\mathfrak{g}^{\mathfrak{h}} \subseteq \mathfrak{g}^Z$ and $[\mathfrak{h}, \mathfrak{g}] \supseteq [Z, \mathfrak{g}]$. By definition however, $\mathfrak{h} \subset \mathcal{Z}(\mathfrak{g}^Z)$ so that $\mathfrak{g}^Z \subseteq \mathfrak{g}^{\mathfrak{h}}$ and both of the previous inclusions are equalities.

Let now $Z' \in \mathfrak{g}$ be such that $\mathfrak{g}^{Z'} = \mathfrak{g}^Z$, then $Z' \in \mathcal{Z}(\mathfrak{g}^{Z'}) = \mathcal{Z}(\mathfrak{g}^Z)$. Since Z' is semisimple, $Z' \in \mathfrak{h}$. It suffices now to notice that, for any $Z' \in \mathfrak{h}$ one has $\mathfrak{g}^{Z'} \supseteq \mathfrak{g}^{\mathfrak{h}} = \mathfrak{g}^Z$ and $[Z', \mathfrak{g}] \subseteq [\mathfrak{h}, \mathfrak{g}] = [Z, \mathfrak{g}]$ with equalities if, and only if, $\alpha(Z') \neq 0$ for all $\alpha \in \Phi$.

(ii) If $\tilde{H} \subset G$ is a torus such that $Z \in \tilde{\mathfrak{h}}_{\text{reg}}$, (i) implies that $\tilde{\mathfrak{h}}_{\text{reg}} \subset \mathfrak{h}_{\text{reg}}$ and therefore that $\tilde{H} \subset H$. \square

3.2. Isomonodromic families of connections. Fix henceforth a torus $H \subset G$. Let P be the holomorphically trivial principal G -bundle over \mathbb{P}^1 and let $\mathcal{U} \subset \mathfrak{h}_{\text{reg}}$ be an open set. Consider a family of connections on P of the form (3), namely

$$\nabla(Z) = d - \left(\frac{Z}{t^2} + \frac{f(Z)}{t} \right) dt$$

where Z varies in \mathcal{U} and the dependence of $f(Z) \in \mathfrak{g}_{\text{od}}$ with respect to Z is arbitrary.

Definition. The family of connections $\nabla(Z)$ is *isomonodromic* if for any $Z_0 \in \mathcal{U}$, there exists a neighborhood $\mathcal{U}_0 \subset \mathcal{U}$ and a ray r such that $\pm r$ are admissible for all $\nabla(Z)$, $Z \in \mathcal{U}_0$ and the Stokes multipliers $S_{\pm}(Z)$ of $\nabla(Z)$ relative to r are constant on \mathcal{U}_0 .

The isomonodromy of the family $\nabla(Z)$ may also be defined as the constancy of the Stokes factors. This requires a little more care since, as pointed out in [5, pg. 190] for example, Stokes rays may split into distinct rays under arbitrarily small deformations of Z . Call a sector $\Sigma \subset \mathbb{C}^*$ *admissible* if its boundary rays are admissible.

Proposition. *The family of connections $\nabla(Z)$ is isomonodromic if, and only if, for any connected open subset $\mathcal{U}_0 \subset \mathcal{U}$ and any convex sector Σ which is admissible for all $\nabla(Z)$, $Z \in \mathcal{U}_0$, the clockwise product*

$$\prod_{\ell \subset \Sigma}^{\curvearrowright} S_\ell(Z)$$

of Stokes factors corresponding to the Stokes rays contained in Σ is constant on \mathcal{U}_0 .

PROOF. This follows from the fact that Stokes factors and multipliers determine each other by Lemma 2.9 and Proposition 5.2. \square

3.3. Isomonodromy equations. The following characterisation of isomonodromic deformations was obtained by Jimbo–Miwa–Ueno [14] when $G = GL_n(\mathbb{C})$ and H is a maximal torus and adapted to the case of a complex, reductive group by Boalch [5, Appendix]. The proof carries over verbatim to the case of an arbitrary algebraic group G and torus $H \subset G$.

Theorem. *Assume that f varies holomorphically in Z . Then, the family of connections $\nabla(Z)$ is isomonodromic if, and only if f satisfies the PDE*

$$df_\alpha = \sum_{\substack{\beta, \gamma \in \Phi: \\ \beta + \gamma = \alpha}} [f_\beta, f_\gamma] d \log \gamma. \quad (13)$$

Remark. The equations (13) form a first order system of *integrable* non-linear PDEs and therefore possess a unique holomorphic solution $f(Z)$ defined in a neighborhood of a fixed $Z_0 \in \mathfrak{h}_{\text{reg}}$ and subject to the initial condition $f(Z_0) = f_0 \in [Z_0, \mathfrak{g}]$.

Remark. Jimbo–Miwa–Ueno and Boalch also give an alternative characterisation of isomonodromy in terms of the existence of a flat connection on $\mathbb{P}^1 \times \mathcal{U}$ which has a logarithmic singularity on the divisor $\{t = \infty\}$ and a pole of order 2 on $\{t = 0\}$, and restricts to $\nabla(Z)$ on each fibre $\{Z\} \times \mathbb{P}^1$. This connection is given by

$$\bar{\nabla} = d - \left[\left(\frac{Z}{t^2} + \frac{f}{t} \right) dt + \sum_{\alpha \in \Phi} f_\alpha \frac{d\alpha}{\alpha} + \frac{dZ}{t} \right].$$

One can check directly that the flatness of this connection is equivalent to (13).

4. THE STOKES MAP

In this section, we express the logarithms of the Stokes factors of the connection ∇ as explicit, universal Lie series in the variables f_α . Using the results of Section 11, we then show how to invert these series to express the f_α as Lie series in the logarithms of the Stokes factors, thus explicitly solving a Riemann–Hilbert problem.

4.1. Completion with respect to finite-dimensional representations. Our formulae for the Stokes factors ∇ are more conveniently expressed inside the completion $\widehat{U\mathfrak{g}}$ of $U\mathfrak{g}$ with respect to the finite-dimensional representations of G . We review below the definition of $\widehat{U\mathfrak{g}}$.

Let Vec be the category of finite-dimensional complex vector spaces and $\text{Rep}(G)$ that of finite-dimensional representations of G . Consider the forgetful functor

$$F : \text{Rep}(G) \rightarrow \text{Vec}.$$

By definition, $\widehat{U\mathfrak{g}}$ is the algebra of endomorphisms of F . Concretely, an element of $\widehat{U\mathfrak{g}}$ is a collection $\Theta = \{\Theta_V\}$, with $\Theta_V \in \text{End}_{\mathbb{C}}(V)$ for any $V \in \text{Rep}(G)$, such that for any $U, V \in \text{Rep}(G)$ and $T \in \text{Hom}_G(U, V)$, the following holds

$$\Theta_V \circ T = T \circ \Theta_U$$

There are natural homomorphisms $G \rightarrow \widehat{U\mathfrak{g}}$ and $U\mathfrak{g} \rightarrow \widehat{U\mathfrak{g}}$ mapping $g \in G$ and $x \in U\mathfrak{g}$ to the elements $\Theta(g), \Theta(x)$ which act on a finite-dimensional representation $\rho : G \rightarrow GL(V)$ as $\rho(g)$ and $\rho(x)$ respectively. The following is well known.

Lemma. *The homomorphisms $G \rightarrow \widehat{U\mathfrak{g}}$ and $U\mathfrak{g} \rightarrow \widehat{U\mathfrak{g}}$ are injective.*

PROOF. The first claim follows immediately from the fact that G has a faithful finite-dimensional representation. For the second, we use the fact that $U\mathfrak{g}$ acts faithfully on $\mathbb{C}[G]$ by left-invariant differential operators. Since this action decreases the degree of polynomials, any $f \in \mathbb{C}[G]$ is contained in a finite-dimensional G -module and the claim follows. \square

We will use the homomorphisms above to think of $U\mathfrak{g}$ as a subalgebra of $\widehat{U\mathfrak{g}}$ and G as a subgroup of the group of invertible elements of $\widehat{U\mathfrak{g}}$ respectively.

4.2. Representing Stokes factors. Fix a Stokes ray ℓ . We show below how to represent the corresponding Stokes factor S_ℓ in two different ways: by elements $\epsilon_\alpha \in \mathfrak{g}_{\text{od}}$ and by elements $\delta_\gamma \in U\mathfrak{g}$.

Consider the subalgebra

$$\mathfrak{n}_\ell = \bigoplus_{\alpha: Z(\alpha) \in \ell} \mathfrak{g}_\alpha \subset \mathfrak{g}.$$

The elements of \mathfrak{n}_ℓ are nilpotent, that is they act by nilpotent endomorphisms on any finite-dimensional representation of G . It follows that the exponential map $\exp: \mathfrak{n}_\ell \rightarrow G$ is an isomorphism onto the unipotent subgroup $N_\ell = \exp(\mathfrak{n}_\ell) \subset G$.

By Proposition 2.6, the Stokes factor S_ℓ lies in N_ℓ . For the first representation of S_ℓ , write

$$S_\ell = \exp \left(\sum_{\alpha: Z(\alpha) \in \ell} \epsilon_\alpha \right) \tag{14}$$

for uniquely defined elements $\epsilon_\alpha \in \mathfrak{g}_\alpha$. For the second, we compute the exponential (14) in $\widehat{U\mathfrak{g}}$ and decompose the result along the weight spaces

$$\widehat{U\mathfrak{g}}_\gamma = \{x \in \widehat{U\mathfrak{g}} \mid \text{ad}(h)x = \gamma(h)x, \forall h \in \mathfrak{h}\}, \gamma \in \mathfrak{h}^*$$

of the adjoint action of \mathfrak{h} . This yields elements $\delta_\gamma \in (U\mathfrak{n}_\ell)_\gamma$ such that

$$S_\ell = 1 + \sum_{\gamma \in \Lambda^Z : Z(\gamma) \in \ell} \delta_\gamma, \quad (15)$$

where $\Lambda^Z \subset \mathfrak{h}^*$ is the lattice generated by Φ^Z and the above identity is to be understood as holding in any finite-dimensional representation of G (where the right-hand side is necessarily finite).

These two representations of S_ℓ are related as follows.

Lemma.

(i) *Let $\gamma \in \Lambda^Z$ be such that $Z(\gamma)$ lies on the Stokes ray ℓ . Then, δ_γ is given by the finite sum*

$$\delta_\gamma = \sum_{n \geq 1} \sum_{\substack{\alpha_i \in \Phi^Z, \\ Z(\alpha_i) \in \ell, \\ \alpha_1 + \dots + \alpha_n = \gamma}} \frac{1}{n!} \epsilon_{\alpha_1} \cdots \epsilon_{\alpha_n}. \quad (16)$$

(ii) *Conversely, let $\alpha \in \Phi^Z$ be such that $Z(\alpha) \in \ell$. Then, ϵ_α is given by the finite sum*

$$\epsilon_\alpha = \sum_{n \geq 1} \sum_{\substack{\gamma_i \in \Lambda^Z, \\ Z(\gamma_i) \in \ell, \\ \gamma_1 + \dots + \gamma_n = \alpha}} \frac{(-1)^{n-1}}{n} \delta_{\gamma_1} \cdots \delta_{\gamma_n}. \quad (17)$$

PROOF. These are the standard expansions of $\exp: \mathfrak{n}_\ell \rightarrow N_\ell$ and $\log: N_\ell \rightarrow \mathfrak{n}_\ell$. \square

4.3. The Stokes map. Since the subsets $\{\alpha \in \Phi : Z(\alpha) \in \ell\}$ partition Φ as ℓ ranges over the Stokes rays of ∇ , we may assemble the elements ϵ_α corresponding to different Stokes rays and form the sum

$$\epsilon = \sum_{\alpha \in \Phi^Z} \epsilon_\alpha \in \bigoplus_{\alpha \in \Phi^Z} \mathfrak{g}_\alpha. \quad (18)$$

We shall refer to the map

$$\mathcal{S}: \bigoplus_{\alpha \in \Phi^Z} \mathfrak{g}_\alpha \longrightarrow \bigoplus_{\alpha \in \Phi^Z} \mathfrak{g}_\alpha \quad (19)$$

mapping f to ϵ as the *Stokes map*. Note that \mathcal{S} depends upon Z .

4.4. The functions M_n . We give below an explicit formula for the Stokes factors of the connection ∇ in terms of iterated integrals. The definition and elementary properties of iterated integrals are reviewed in Section 7.

Definition. Set $M_1(z_1) = 2\pi i$ and, for $n \geq 2$, define the function $M_n: (\mathbb{C}^*)^n \rightarrow \mathbb{C}$ by the iterated integral

$$M_n(z_1, \dots, z_n) = 2\pi i \int_C \frac{dt}{t - s_1} \circ \cdots \circ \frac{dt}{t - s_{n-1}},$$

where $s_i = z_1 + \cdots + z_i$, $1 \leq i \leq n$ and the path of integration C is the line segment $(0, s_n)$, perturbed if necessary to avoid any point $s_i \in [0, s_n]$ by small clockwise arcs.

Remark. The iterated integrals defining the functions M_n are convergent by Lemma 7.4 below since the assumption that $z_i \in \mathbb{C}^*$ for all i implies in particular that $s_1 \neq 0$ and $s_{n-1} \neq s_n$.

4.5. Formula for the Stokes factors. The following result will be proved in Section 10.

Theorem. *The Stokes factor S_ℓ corresponding to the ray ℓ is given by*

$$S_\ell = 1 + \sum_{n \geq 1} \sum_{\substack{\alpha_1, \dots, \alpha_n \in \Phi^Z \\ Z(\alpha_1 + \cdots + \alpha_n) \in \ell}} M_n(Z(\alpha_1), \dots, Z(\alpha_n)) f_{\alpha_1} \cdots f_{\alpha_n} \quad (20)$$

where the equality is understood as holding in any finite-dimensional representation of G and the sum over n is absolutely convergent.

In terms of weight components, (20) reads

$$\delta_\gamma = \sum_{n \geq 1} \sum_{\substack{\alpha_1, \dots, \alpha_n \in \Phi^Z \\ \alpha_1 + \cdots + \alpha_n = \gamma}} M_n(Z(\alpha_1), \dots, Z(\alpha_n)) f_{\alpha_1} f_{\alpha_2} \cdots f_{\alpha_n}. \quad (21)$$

for any $\gamma \in \Lambda^Z$ such that $Z(\gamma) \in \ell$.

Remark. Theorem 4.5 shows that the Stokes factors of ∇ are given by periods. Their appearance in this context stems from the fact that their computation reduces, via the results of Sections 8–9, to one of partial monodromies of the Fourier–Laplace transform $\widehat{\nabla}$ of ∇ , which are well-known to be given by iterated integrals. It seems an interesting problem to determine whether the Stokes factors of a connection with arbitrary order poles on \mathbb{P}^1 are also expressible in terms of explicit periods.

4.6. The functions L_n . We next state a formula for the Stokes map giving the element ϵ in terms of f . We first define the special functions appearing in this formula.

Definition. The function $L_n : (\mathbb{C}^*)^n \rightarrow \mathbb{C}$ is given by $L_1(z_1) = 2\pi i$ and, for $n \geq 2$,

$$L_n(z_1, \dots, z_n) = \sum_{k=1}^n \sum_{\substack{0=i_0 < \cdots < i_k=n \\ s_{i_j} - s_{i_{j-1}} \in \mathbb{R}_{>0} \cdot s_n}} \frac{(-1)^{k-1}}{k} \prod_{j=0}^{k-1} M_{i_{j+1}-i_j}(z_{i_{j+1}}, \dots, z_{i_{j+1}}),$$

where $s_j = z_1 + \cdots + z_j$.

Remark. Note that $L_1 \equiv M_1$ and that on the open subset

$$(z_1, \dots, z_n) \in (\mathbb{C}^*)^n \text{ such that } s_i \notin [0, s_n] \text{ for } 0 < i < n$$

the inner sum above is empty unless $k = 1$ so that $L_n(z_1, \dots, z_n) = M_n(z_1, \dots, z_n)$. Thus L_n agrees with M_n on the open subset where it is holomorphic and differs from it by how it has been extended onto the cutlines.

The functions L_n are more complicated to define than the functions M_n . Unlike the latter however, they give rise to Lie series by Theorem 4.7 (i).

Remark. For $n \geq 2$, the function L_n satisfies

$$L_n(z_1, \dots, z_n) = 0 \quad \text{if} \quad z_1 + \dots + z_n = 0 \quad (22)$$

Indeed, the summation condition above becomes $z_{i_j} + \dots + z_{i_{j-1}+1} = 0$ for any $j = 1, \dots, n$ and $M_m(w_1, \dots, w_m) = 0$ whenever $w_1 + \dots + w_m = 0$.

4.7. Formula for the Stokes map.

Theorem.

- (i) Let x_1, \dots, x_m be elements in a Lie algebra \mathcal{L} . For any $(z_1, \dots, z_m) \in (\mathbb{C}^*)^m$, the finite sum

$$\sum_{\sigma \in \text{Sym}_m} L_m(z_{\sigma(1)}, \dots, z_{\sigma(m)}) x_{\sigma(1)} \cdots x_{\sigma(m)} \quad (23)$$

is a Lie polynomial in x_1, \dots, x_m and therefore lies in $\mathcal{L} \subset U\mathcal{L}$.

- (ii) The element $\epsilon = \mathcal{S}(f)$ is given by the following Lie series in the variables $\{f_\alpha\}_{\alpha \in \Phi^Z}$

$$\epsilon_\alpha = \sum_{n \geq 1} \sum_{\substack{\alpha_i \in \Phi^Z \\ \alpha_1 + \dots + \alpha_n = \alpha}} L_n(Z(\alpha_1), \dots, Z(\alpha_n)) f_{\alpha_1} f_{\alpha_2} \cdots f_{\alpha_n} \quad (24)$$

where the sum over n is absolutely convergent.

PROOF. Substituting (21) into (17) shows that (24) holds in $\widehat{U\mathfrak{g}}$, where the sum over n is absolutely convergent in any finite-dimensional representation of G .

To prove (i), we may assume that $z_1 + \dots + z_m \neq 0$ since $L_m(z_1, \dots, z_m) = 0$ if $z_1 + \dots + z_m = 0$. We need to show that if \mathcal{L} is the free Lie algebra generated by x_1, \dots, x_m , the sum (23) lies in \mathcal{L} . \mathcal{L} is $\mathbb{Z}_{\geq 0}$ -graded by setting $\deg(x_i) = 1$ and it suffices to prove this for the finite-dimensional quotient $\mathcal{L}_{\leq m}$ of \mathcal{L} by elements of degree $\geq m + 1$.

We shall do so by applying the version of (ii) just obtained to the semi-direct product $B = T \ltimes L_{\leq m}$ of the algebraic torus $T = (\mathbb{C}^*)^m$ by the unipotent group $L_{\leq m}$ with Lie algebra $\mathcal{L}_{\leq m}$, where T acts on $\mathcal{L}_{\leq m}$ by $(w_1, \dots, w_m) x_i = w_i x_i$. Let $\mathfrak{t} = \mathbb{C}^m$ be the Lie algebra of T , $\{E_i\}_{i=1}^m$ its canonical basis and $\{\theta_i\}$ the corresponding dual basis of \mathfrak{t}^* . The root system $\Phi(B, T) \subset \mathfrak{t}^*$ is contained in the set of elements of the form $\sum_i p_i \theta_i$, with $p_i \in \mathbb{Z}_{\geq 0}$ and $\sum_i p_i \leq m$.

Consider the connection ∇ with structure group B and coefficients $Z \in \mathfrak{t}$ and $f \in \mathcal{L}_{\leq m}$ given by

$$Z = z_1 E_1 + \dots + z_m E_m \quad \text{and} \quad f = x_1 + \dots + x_m.$$

The logarithms of the Stokes factors of ∇ determine an element $\epsilon \in \mathcal{L}_{\leq m}$ whose components are given by (24), understood as an identity in $\widehat{U\mathcal{L}_{\leq m}}$. Since in this case the sum over n is finite, (24) actually holds as an identity in $U\mathcal{L}_{\leq m}$.

Note that the roots $\theta_1, \dots, \theta_m$ and $\theta_1 + \dots + \theta_m$ lie in Φ^Z since, by assumption $Z(\theta_i) = z_i \neq 0$ and $Z(\theta_1 + \dots + \theta_m) = z_1 + \dots + z_m \neq 0$. Since the only non-zero components of f are $f_{\theta_i} = x_i$, $i = 1, \dots, m$, the component $\epsilon_{\theta_1 + \dots + \theta_m}$ is precisely equal to (23). This proves (i), and thus that (24) holds as an identity in $\mathfrak{g} \subset \widehat{U\mathfrak{g}}$. \square

Remark. The right-hand side of (21) makes sense for any $\gamma \in \Lambda^Z$. It is easy to show using Theorem 4.7 that each summand in n is equal to zero unless γ is such that $Z(\gamma)$ lies on a Stokes ray of ∇ . Thus, the identity (21) holds for any $\gamma \in \Lambda^Z$.

Similarly, (24) holds for any $\alpha \in \Phi$ since, for $\alpha \in \Phi \sqcup \{0\} \setminus \Phi^Z$, the left-hand side is equal to zero by definition and the right-hand side vanishes by (22). Thus, (24) may equivalently be written as

$$\epsilon = \sum_{n \geq 1} \sum_{\alpha_1, \dots, \alpha_n \in \Phi} L_n(Z(\alpha_1), \dots, Z(\alpha_n)) f_{\alpha_1} f_{\alpha_2} \cdots f_{\alpha_n}$$

4.8. Inverse of the Stokes map. By Theorem 4.7 and [18, Thm. 1.5.6], the series (24) converges uniformly on compact subsets of $\mathfrak{g}_{\text{od}} = \bigoplus_{\alpha \in \Phi^Z} \mathfrak{g}_\alpha$. Thus, the Stokes map $\mathcal{S} : \mathfrak{g}_{\text{od}} \rightarrow \mathfrak{g}_{\text{od}}$ is holomorphic, satisfies $\mathcal{S}(0) = 0$ and its differential at $f = 0$ is invertible since L_1 is identically equal to $2\pi i$. By the inverse function theorem, \mathcal{S} possesses an analytic inverse \mathcal{S}^{-1} defined on a neighborhood of $\epsilon = 0$.

Theorem. *The Taylor series of \mathcal{S}^{-1} at $\epsilon = 0$ is given by a Lie series in the variables $\{\epsilon_\alpha\}_{\alpha \in \Phi^Z}$ of the form*

$$f_\alpha = \sum_{n \geq 1} \sum_{\substack{\alpha_i \in \Phi^Z \\ \alpha_1 + \cdots + \alpha_n = \alpha}} J_n(Z(\alpha_1), \dots, Z(\alpha_n)) \epsilon_{\alpha_1} \epsilon_{\alpha_2} \cdots \epsilon_{\alpha_n} \quad (25)$$

where the functions $J_n : (\mathbb{C}^*)^n \rightarrow \mathbb{C}$ are independent of \mathfrak{g} and such that $J_1 \equiv 1/2\pi i$ and, for $n \geq 2$,

$$J_n(z_1, \dots, z_n) = 0 \quad \text{if} \quad z_1 + \cdots + z_n = 0 \quad (26)$$

Remark. Analogously to Remark 4.7, (25) may equivalently be written as

$$f = \sum_{n \geq 1} \sum_{\alpha_1, \dots, \alpha_n \in \Phi} J_n(Z(\alpha_1), \dots, Z(\alpha_n)) \epsilon_{\alpha_1} \epsilon_{\alpha_2} \cdots \epsilon_{\alpha_n}$$

4.9. The functions J_n . Theorem 4.8 will be proved in Section 11 by formally inverting the power series (24). The functions J_n are explicitly described there as sums of products of the functions L_n indexed by plane rooted trees. For example,

$$(2\pi i)^3 J_3(z_1, z_2, z_3) = L_2(z_1 + z_2, z_3) L_2(z_1, z_2) - L_3(z_1, z_2, z_3) + L_2(z_1, z_2 + z_3) L_2(z_2, z_3)$$

corresponding to the three distinct plane rooted trees with 3 leaves.

Theorem. *The function $J_n : (\mathbb{C}^*)^n \rightarrow \mathbb{C}$ is continuous and holomorphic on the complement of the hyperplanes*

$$H_{ij} = \{z_i + \cdots + z_j = 0\}, \quad 1 \leq i < j \leq n$$

in the domain

$$\mathcal{D}_n = \{(z_1, \dots, z_n) \in (\mathbb{C}^*)^n \mid z_i/z_{i+1} \notin \mathbb{R}_{>0} \text{ for } 1 \leq i < n\} \quad (27)$$

Moreover, it satisfies the differential equation

$$dJ_n(z_1, \dots, z_n) = \sum_{i=1}^{n-1} J_i(z_1, \dots, z_i) J_{n-i}(z_{i+1}, \dots, z_n) d \log \left(\frac{z_{i+1} + \cdots + z_n}{z_1 + \cdots + z_i} \right) \quad (28)$$

PROOF. The stated properties of the functions J_n will be obtained by applying Theorem 4.8 to the group $B \subset GL_{n+1}(\mathbb{C})$ of upper triangular matrices. Choose as torus $H \subset B$ the subgroup of diagonal matrices, let $N \subset B$ be the subgroup of strictly upper triangular matrices and denote their Lie algebras by $\mathfrak{b}, \mathfrak{h}$ and \mathfrak{n} respectively. Let $\{e_i\}_{i=1}^{n+1}$ be the canonical basis of \mathbb{C}^{n+1} , $E_{ij}e_k = \delta_{jk}e_i$ the corresponding elementary matrices, and $\{\theta_i\}_{i=1}^{n+1}$ the basis of \mathfrak{h}^* given by $\theta_i(E_{jj}) = \delta_{ij}$. The root system $\Phi(B, H)$ consists of the linear forms $\alpha = \theta_i - \theta_j$, $1 \leq i < j \leq n+1$. Note that in this case the sum (25) is finite and therefore defines, for any fixed $Z \in \mathfrak{h}$, a global inverse to the Stokes map $\mathcal{S} : [Z, \mathfrak{g}] \rightarrow [Z, \mathfrak{g}]$.

Set

$$\mathcal{D}'_n = \mathcal{D}_n \setminus \bigcup_{1 \leq i < j \leq n} H_{ij}$$

Let $Z \in \mathfrak{h}$ be given by

$$Z = \text{diag}(z_1 + \cdots + z_n, z_2 + \cdots + z_n, \dots, z_n, 0)$$

Then $Z(\theta_i - \theta_j) = z_i + \cdots + z_{j-1}$ so that $Z \in \mathfrak{h}_{\text{reg}}$ if, and only if $(z_1, \dots, z_n) \notin \bigcup_{i \neq j} H_{ij}$.

Let $\epsilon = \sum_{i=1}^n E_{i,i+1} \in \mathfrak{n}$. Since $z_i \neq 0$, ϵ lies in

$$[Z, \mathfrak{b}] = \bigoplus_{\substack{1 \leq i < j \leq n+1 \\ z_i + \cdots + z_{j-1} \neq 0}} \mathbb{C}E_{ij}$$

and therefore defines Stokes data for a unique B -connection of the form (3). The corresponding Stokes rays are $\mathbb{R}_{>0}(z_i + \cdots + z_{j-1})$ and include in particular the rays $\ell_i = \mathbb{R}_{>0}z_i$. Since $\epsilon_\alpha = 0$ unless $\alpha = \theta_i - \theta_{i+1}$, the only non-trivial Stokes factors correspond to the rays ℓ_i and are given by

$$S_{\ell_i} = \exp \left(\sum_{j: z_j \in \ell_i} E_{j,j+1} \right)$$

Set $S_i = \exp(E_{i,i+1}) = 1 + E_{i,i+1}$. Given that $[E_{i,i+1}, E_{j,j+1}] = 0$ if $|i - j| \geq 2$, the above Stokes factors are given by $S_{\ell_i} = \prod_{j: z_j \in \ell_i} S_j$ since $\ell_i \neq \ell_{i \pm 1}$ on \mathcal{D}'_n . Thus, varying (z_1, \dots, z_n) in \mathcal{D}'_n and keeping ϵ fixed yields an isomonodromic family of connections.

Since the Stokes map $\mathcal{S} : [Z, \mathfrak{g}] \rightarrow [Z, \mathfrak{g}]$ has a global inverse for $G = B$, it follows by Remark 3.3 that the corresponding $f = f(Z)$ varies holomorphically in $Z \in \mathfrak{h}_{\text{reg}}$ and satisfies the isomonodromy equations (13). Computing now (25) in the vector representation yields

$$f_{\theta_i - \theta_j} = J_{j-i}(z_i, \dots, z_{j-1})E_{ij}$$

for any $i < j$. The claimed regularity of the functions J_i , $i \leq n$ on \mathcal{D}'_n and the differential equation (28) now follow from that of f and the equations (13). \square

Remark. By arguing as in [15, Prop. 3.10], it is easy to prove by induction on n , using the PDE (28) that the function J_n possesses a holomorphic extension \tilde{J}_n

to \mathcal{D}_n . Although we have not checked this in full, we believe that this extension coincides with our combinatorial definition of J_n .

4.10. Irregular Riemann–Hilbert correspondence. It is important to distinguish the Stokes map (19) from a related map also studied by Balsler–Jurkat–Lutz [1] and Boalch [6, 7]. Rather than studying connections of the form (3), one can consider instead meromorphic connections on the trivial principal G -bundle over the unit disc $D \subset \mathbb{C}$ which have the form

$$d - \left(\frac{Z}{t^2} + \frac{f(t)}{t} \right) dt \quad (29)$$

where $f: D \rightarrow \mathfrak{g}$ is holomorphic.

One can define Stokes data for such a connection as in Section 2.7, and consider the map sending the set of gauge equivalence classes of such connections to that of possible Stokes data. Boalch refers to this map as the *irregular Riemann–Hilbert map*. Extending results of [1] for $G = GL_n(\mathbb{C})$, he shows that, for G reductive and Z regular semisimple, this map is in fact an isomorphism.

In contrast, as shown in [16], the Stokes map \mathcal{S} is neither injective nor surjective in general, even when $G = GL_2(\mathbb{C})$. Put another way, not every connection of the form (29) can be put into the constant-coefficient form (3) by a gauge transformation, and even when that is possible, the resulting connection (3) is not in general unique.

It follows from Theorem 4.8 that whenever the sum (25) is absolutely convergent over n , it successfully inverts the Stokes map \mathcal{S} , in that the connection (3) determined by $f \in \mathfrak{g}_{\text{od}}$ has Stokes factors given by (14). In spite of our assumption that $f \in \mathfrak{g}_{\text{od}}$, which does not hold in the counterexamples of [16], we do not expect the Stokes map to be bijective and therefore the sum to be absolutely convergent over n for arbitrary ϵ .

5. STOKES MULTIPLIERS

Throughout this section, we fix a ray $r = \mathbb{R}_{>0}e^{i\pi\theta}$ such that $\pm r$ are admissible for the connection (3) and consider the Stokes multipliers S_{\pm} relative to r . We shall give an explicit formula for S_{\pm} analogous to that for the Stokes factors given by Theorem 4.5.

5.1. Representing Stokes multipliers. We first show how to represent S_{\pm} by an element $\kappa \in \widehat{U\mathfrak{g}}$. Let $\pm i\mathbb{H}_r$ be the connected components of $\mathbb{C} \setminus \mathbb{R}e^{i\pi\theta}$. These determine a partition of $\Phi^Z = \Phi_+^Z \sqcup \Phi_-^Z$ given by

$$\Phi_{\pm}^Z = \{\alpha \in \Phi^Z : Z(\alpha) \in \pm i\mathbb{H}_r\}$$

Let $\Lambda_{\pm}^Z \subset \mathfrak{h}^* \setminus \{0\}$ be the cones spanned by the linear combinations of elements in Φ_{\pm}^Z with coefficients in $\mathbb{N}_{>0}$. Similarly to §4.2, it follows from Proposition 2.6 that there is a unique element

$$\kappa = \sum_{\gamma \in \Lambda_+^Z \sqcup \Lambda_-^Z} \kappa_{\gamma} \in \widehat{U\mathfrak{g}}$$

such that the Stokes multipliers S_{\pm} are respectively equal to

$$S_+ = 1 + \sum_{\gamma \in \Lambda_+^Z} \kappa_{\gamma} \quad \text{and} \quad (S_-)^{-1} = 1 + \sum_{\gamma \in \Lambda_-^Z} \kappa_{\gamma},$$

5.2. Given $\gamma \in \Lambda_+^Z$, set

$$\phi(\gamma) = \frac{1}{\pi} \arg Z(\gamma) \in (\theta, \theta + 1).$$

The following result gives the relation between the elements κ and δ .

Proposition.

(i) For all $\gamma \in \Lambda_+^Z$, there is a finite sum

$$\kappa_{\gamma} = \sum_{n \geq 1} \sum_{\substack{\gamma_1 + \dots + \gamma_n = \gamma \\ \phi(\gamma_1) > \dots > \phi(\gamma_n)}} \delta_{\gamma_1} \cdots \delta_{\gamma_n}, \quad (30)$$

where the sum is over elements $\gamma_i \in \Lambda_+^Z$.

(ii) Conversely, for $\gamma \in \Lambda_+^Z$

$$\delta_{\gamma} = \sum_{n \geq 1} \sum_{\substack{\gamma_1 + \dots + \gamma_n = \gamma \\ \phi(\gamma_1 + \dots + \gamma_i) > \phi(\gamma)}} (-1)^{n-1} \kappa_{\gamma_1} \cdots \kappa_{\gamma_n}, \quad (31)$$

PROOF. (i) follows from substituting (15) into the formula of Lemma 2.9. (ii) follows from Reineke's inversion of formula (30) [17, Section 5]. \square

Remark. The operation of replacing the ray r by the opposite ray $-r$ exchanges Λ_+^Z and Λ_-^Z and changes the Stokes multipliers (S_+, S_-) to (S_-^{-1}, S_+^{-1}) thus leaving the element κ unchanged. This gives an easy way to obtain similar expressions to (30) and (31) for the case $\gamma \in \Lambda_-^Z$.

Remark. As pointed out in §2.7, the Stokes multipliers of ∇ are determined by the Stokes factors via Lemma 2.9. Conversely, it is well-known that, for G reductive at least, the Stokes factors can be recovered from the Stokes multipliers [1, Lemma 2], [7]. To the best of our knowledge however, no explicit formula was known for this procedure, even in the case of $GL_n(\mathbb{C})$. Reineke's inversion formula (31) gives such a formula.

5.3. **The functions Q_n .** We give below a formula for the Stokes multiplier S_+ . The special functions $Q_n(z_1, \dots, z_n)$ appearing in this formula have the property that

$$Q_n(z_1, \dots, z_n) = 0$$

unless $\pm s_n \in i\mathbb{H}_r$, where $s_n = z_1 + \dots + z_n$. Moreover, $Q_n(z_1, \dots, z_n)$ is invariant under the operation of changing r to $-r$. Thus, it is enough to define $Q_n(z_1, \dots, z_n)$ when $s_n \in i\mathbb{H}_r$.

Definition. Set $Q_1 \equiv 2\pi i$. For $n \geq 2$ and $(z_1, \dots, z_n) \in (\mathbb{C}^*)^n$ such that $s_n \in i\mathbb{H}_r$, define

$$Q_n(z_1, \dots, z_n) = 2\pi i \int_C \frac{dt}{t - s_1} \circ \dots \circ \frac{dt}{t - s_{n-1}}$$

where the path C starts at 0, goes out along the ray $-r$ avoiding any points s_i by small clockwise loops, goes clockwise round a large circle, and finally comes back along the ray $s_n - r$, again avoiding any points s_j by small anticlockwise loops, to finish at the point s_n .

5.4. Formula for the Stokes multipliers.

Theorem. *If r is a ray such that $\pm r$ are admissible, the components κ_γ of the Stokes multiplier S_+ of γ are given by the sum*

$$\kappa_\gamma = \sum_{n \geq 1} \sum_{\substack{\alpha_i \in \Phi^Z \\ \alpha_1 + \dots + \alpha_n = \gamma}} Q_n(Z(\alpha_1), \dots, Z(\alpha_n)) f_{\alpha_1} f_{\alpha_2} \dots f_{\alpha_n}$$

which is absolutely convergent over n .

PROOF. Let $i\overline{\mathbb{H}}_r$ be the semi-closed half-plane $i\mathbb{H}_r \sqcup -r$. Note first that the sum in (30) may be taken over all $\gamma_i \in \Lambda$ such that $Z(\gamma_i) \in i\overline{\mathbb{H}}_r$ since, for such γ , $\delta_\gamma = 0$ unless $\gamma \in \Lambda_+^Z$.

Substituting (21) in (30) and using Remark 4.7 yields, for any $\gamma \in \Lambda_+^Z$,

$$\kappa_\gamma = \sum_{n \geq 1} \sum_{\substack{\alpha_1, \dots, \alpha_n \in \Phi^Z \\ \alpha_1 + \dots + \alpha_n = \gamma}} \tilde{Q}_n(Z(\alpha_1), \dots, Z(\alpha_n)) f_{\alpha_1} \dots f_{\alpha_n}$$

where for $z_1, \dots, z_n \in \mathbb{C}^*$ such that $s_n \in i\mathbb{H}_r$, $\tilde{Q}_n(z_1, \dots, z_n)$ is defined by

$$\tilde{Q}_n(z_1, \dots, z_n) = \sum_{\substack{1 \leq k \leq n \\ 0 = i_0 < \dots < i_k = n \\ s_{i_1}, s_{i_2} - s_{i_1}, \dots, s_{i_k} - s_{i_{k-1}} \in i\overline{\mathbb{H}}_r \\ \phi(s_{i_1}) > \phi(s_{i_2} - s_{i_1}) > \dots > \phi(s_{i_k} - s_{i_{k-1}})}} \prod_{j=0}^{k-1} M_{i_{j+1} - i_j}(z_{i_{j+1}}, \dots, z_{i_{j+1}}) \quad (32)$$

with $s_j = z_1 + \dots + z_j$ and $\phi(z) = \frac{1}{\pi} \arg(z) \in (\theta, \theta + 1)$. The result now follows from the Lemma below. \square

Lemma. *For any $z_1, \dots, z_n \in \mathbb{C}^*$ such that $s_n \in i\mathbb{H}_r$,*

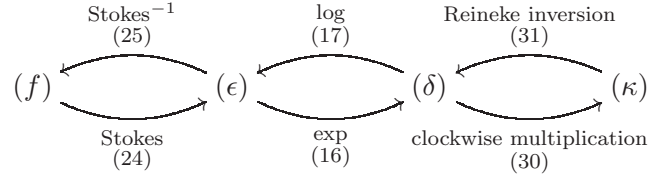
$$\tilde{Q}_n(z_1, \dots, z_n) = Q_n(z_1, \dots, z_n)$$

PROOF. To prove this claim consider the path C of Definition 5.3 as a piece of string and the points s_i for $1 \leq i \leq n-1$ as pegs. Tightening the string will give a convex polygon with vertices some subset of the s_i . Suppose s_p is the first peg. Then applying Corollary 7.6 we can move the string inside this first peg at the expense of adding a term

$$M_p(z_1, \dots, z_p) Q_{n-p}(z_{p+1}, \dots, z_n).$$

By induction, $Q_{n-p}(z_{p+1}, \dots, z_n)$ is a sum over convex polygons so we obtain the part of the sum on the right-hand side of (32) corresponding to $i_1 = p$. Tightening the string again it catches on another peg and we repeat. \square

5.5. The following diagram summarizes the relationships between the elements δ, ϵ representing the Stokes factors, the element κ representing the Stokes multipliers, and the element $f \in \mathfrak{g}_{\text{od}}$ defining ∇ .



6. PROOF OF PROPOSITION 2.6

6.1. Let r be a ray and $\mathbb{H}_r \subset \mathbb{C}^*$ the open half-plane given by

$$\mathbb{H}_r = \{uv \mid u \in r, \text{Re}(v) > 0\} \quad (33)$$

Lemma. *Let $\lambda \in \mathbb{C}^*$ be a non-zero complex number. Then, the function $e^{-\lambda/t}$ has a limit $L \in \mathbb{C}$ as $t \rightarrow 0$ along the ray r if, and only if $\lambda \in \mathbb{H}_r$, in which case $L = 0$.*

PROOF. Write $\lambda = \rho e^{i\theta}$ and $t = \sigma e^{i\phi}$. Then,

$$e^{-\lambda/t} = e^{-\frac{\rho}{\sigma} e^{i(\theta-\phi)}} = e^{-\frac{\rho}{\sigma} \cos(\theta-\phi)} e^{-i\frac{\rho}{\sigma} \sin(\theta-\phi)}$$

This has a finite limit as $\sigma \rightarrow 0$ if, and only if $\theta \in (\phi - \frac{\pi}{2}, \phi + \frac{\pi}{2})$ and, in that case, decreases exponentially to 0. \square

6.2. Let U be a finite-dimensional vector space and $Z \in \text{End}(U)$ a semisimple endomorphism of U . Let $\sigma(Z) \subset \mathbb{C}$ be the set of eigenvalues of Z and

$$U = \bigoplus_{\lambda \in \sigma(Z)} U_\lambda \quad (34)$$

the corresponding decomposition of U into eigenspaces of Z .

Lemma. *Let r_1, r_2 be two rays such that $r_1 \neq -r_2$ and $u \in U$ an element such that*

$$e^{-Z/t} u \rightarrow 0 \quad \text{as } t \rightarrow 0 \text{ in } \mathbb{H}_{r_1} \cap \mathbb{H}_{r_2}$$

Then,

$$u \in \bigoplus_{\lambda \in \overline{\Sigma}} U_\lambda$$

where $\overline{\Sigma} \subset \mathbb{C}^*$ is the closed convex sector bounded by r_1 and r_2 .

PROOF. Let $u = \sum_{\lambda} u_{\lambda}$ be the decomposition of u corresponding to (34). Since each U_{λ} is stable under $\exp(-Z/t)$ and $e^{-Z/t}u_{\lambda} = e^{-\lambda/t}u_{\lambda}$, we find that $u_0 = 0$ and that $e^{-\lambda/t} \rightarrow 0$ as $t \rightarrow 0$ in $\mathbb{H}_{r_1} \cap \mathbb{H}_{r_2}$ for any λ such that $u_{\lambda} \neq 0$. Applying Lemma 6.1 to a ray r contained in $\mathbb{H}_{r_1} \cap \mathbb{H}_{r_2}$ then shows that any such λ is contained in

$$\bigcap_{r \subset \mathbb{H}_{r_1} \cap \mathbb{H}_{r_2}} \mathbb{H}_r = \overline{\Sigma}$$

□

6.3. Let G be an affine algebraic group and \mathfrak{g} its Lie algebra. Let $Z \in \mathfrak{g}$ be a semisimple element and decompose \mathfrak{g} as the sum $\bigoplus_{\lambda \in \mathbb{C}} \mathfrak{g}_{\lambda}$ of eigenspaces for the adjoint action of Z . The following result is Proposition 2.6 of Section 2. It was proved by Boalch [7, lemma 6] in the case where G is reductive and Z is regular by using the Bruhat decomposition of G .

Proposition. *Let r_1, r_2 be two rays such that $r_1 \neq -r_2$ and $g \in G$ an element such that*

$$e^{-Z/t} \cdot g \cdot e^{Z/t} \rightarrow 1 \quad \text{as } t \rightarrow 0 \quad \text{in } \mathbb{H}_{r_1} \cap \mathbb{H}_{r_2}$$

Then g is unipotent and $X = \log(g)$ lies in

$$\bigoplus_{\lambda \in \overline{\Sigma}} \mathfrak{g}_{\lambda}$$

where $\overline{\Sigma} \subset \mathbb{C}^$ is the closed convex sector bounded by r_1 and r_2 .*

PROOF. Embed G as a closed subgroup of $GL(V)$, where V is a faithful representation. Applying Lemma 6.2 to $u = (g - 1) \in \mathfrak{gl}(V) = U$, we find that u lies in the span of the $\text{ad}(Z)$ -eigenspaces of $\mathfrak{gl}(V)$ corresponding to eigenvalues lying in $\overline{\Sigma}$. In particular, u is a nilpotent endomorphism of V . The finite sum

$$X = \text{Log}(g) = \sum_{n \geq 1} (-1)^{n-1} \frac{u^n}{n}$$

is then a well-defined element of $\mathfrak{gl}(V)$ and lies in $\mathfrak{g} \subset \mathfrak{gl}(V)$ because $g \in G$. Since the $\text{ad}(Z)$ -eigenvalues of u^n are contained in the n -fold sum $\overline{\Sigma} + \cdots + \overline{\Sigma} \subset \overline{\Sigma}$, the result follows. □

7. FUCHSIAN CONNECTIONS AND MULTILOGARITHMS

This section contains some basic results about computing parallel transport for Fuchsian connections using iterated integrals. These results are presumably well-known to experts but we failed to find a suitable reference.

7.1. For an introduction to iterated integrals see for example [10]. We start by recalling their definition. Let $\omega_1, \dots, \omega_n$ be 1-forms defined on a domain $U \subset \mathbb{C}$, and $\gamma: [0, 1] \rightarrow U$ a path in U . Let

$$\Delta = \{(t_1, \dots, t_n) \in [0, 1]^n : 0 \leq t_1 \leq \dots \leq t_n \leq 1\} \subset [0, 1]^n$$

be the unit simplex. By definition,

$$\int_{\gamma} \omega_1 \circ \dots \circ \omega_n = \int_{\Delta} f_1(t_1) \cdots f_n(t_n) dt_1 \cdots dt_n$$

where $\gamma^* \omega_i = f_i(t) dt$. The following is easily checked.

Lemma.

(i) Let $\bar{\gamma}(t) = \gamma(1 - t)$ be the opposite path to γ . Then,

$$\int_{\bar{\gamma}} \omega_1 \circ \dots \circ \omega_n = \int_{\gamma} \omega_n \circ \dots \circ \omega_1$$

(ii) Let $\phi: \mathbb{C} \rightarrow \mathbb{C}$ be a smooth map, then

$$\int_{\phi \circ \gamma} \omega_1 \circ \dots \circ \omega_n = \int_{\gamma} \phi^* \omega_1 \circ \dots \circ \phi^* \omega_n$$

There is an alternative convention obtained by using the simplex

$$\Delta^* = \{(t_1, \dots, t_n) \in [0, 1]^n : 1 \geq t_1 \geq \dots \geq t_n \geq 0\} \subset [0, 1]^n$$

instead of Δ . We denote the resulting integral with a $*$ above the integral sign. Thus,

$$\int_{\gamma}^* \omega_1 \circ \dots \circ \omega_n = \int_{\Delta^*} f_1(t_1) \cdots f_n(t_n) dt_1 \cdots dt_n$$

This convention is the more natural one for computing parallel transport and is the one we shall use in this section. On the other hand, the convention relying on the simplex Δ seems to be the preferred one in the study of multilogarithms [9, 10]. It is of course easy to translate between these conventions since the change of variables $t_i^* = t_{n+1-i}$ yields

$$\int_{\gamma}^* \omega_1 \circ \dots \circ \omega_n = \int_{\gamma} \omega_n \circ \dots \circ \omega_1 \tag{35}$$

7.2. Let V be a finite-dimensional vector space and let $\mathcal{P} \subset \mathbb{C}$ be a finite set of points. Given a choice of residue $A_p \in \text{End}(V)$ for each $p \in \mathcal{P}$ we can define a meromorphic connection on the trivial vector bundle over \mathbb{P}^1 with fibre V by writing

$$\widehat{\nabla} = d - \sum_{p \in \mathcal{P}} \frac{A_p}{z - p} dz.$$

Suppose given a smooth path $\gamma: [0, 1] \rightarrow \mathbb{C} \setminus \mathcal{P}$. The parallel transport of $\widehat{\nabla}$ along γ is the invertible linear map $\text{PT}_{\gamma} \in \text{GL}(V)$ obtained by analytically continuing flat sections of $\widehat{\nabla}$ along γ . Thus, if Φ is a fundamental solution defined near $\gamma(0)$, then

$$\text{PT}_{\gamma} = \Phi(\gamma(1)) \cdot \Phi(\gamma(0))^{-1}.$$

where $\Phi(\gamma(1))$ is the value at $\gamma(1)$ of the analytic continuation of Φ along γ .

Solving the differential equation for flat sections of $\widehat{\nabla}$ using Picard iteration gives the following power series expansion for such parallel transport maps (see, *e.g.* [10, Lemma 2.5]).

Theorem. *For any smooth path $\gamma: [0, 1] \rightarrow \mathbb{C} \setminus \mathcal{P}$ one has*

$$\text{PT}_\gamma = 1 + \sum_{\substack{n \geq 1 \\ p_1, \dots, p_n \in \mathcal{P}}} I_{\gamma, n}(p_1, \dots, p_n) A_{p_1} \cdots A_{p_n},$$

where the sum is absolutely convergent, and the coefficients are iterated integrals

$$I_{\gamma, n}(z_1, \dots, z_n) = \int_\gamma^* \frac{dz}{z - z_1} \circ \cdots \circ \frac{dz}{z - z_n}.$$

7.3. We assume for the rest of the section that each of the residues A_p is nilpotent. In particular the connection $\widehat{\nabla}$ is *non-resonant*, that is the eigenvalues of the residues A_p do not differ by positive integers. In this situation it is well-known that for any connected and simply-connected neighbourhood U_p of a pole $p \in \mathcal{P}$ there is a unique holomorphic function $H_p: U_p \rightarrow GL(V)$ with $H_p(p) = 1$ such that for any determination of the function $\log(z - p)$, the multivalued holomorphic function

$$\Phi_p(z) = H_p(z)(z - p)^{A_p} = H_p(z) \exp(A_p \log(z - p)), \quad (36)$$

is a fundamental solution of $\widehat{\nabla}$. For details see, *e.g.* [13]. We shall refer to Φ_p as the canonical fundamental solution of $\widehat{\nabla}$ relative to a chosen determination of $\log(z - p)$.

Proposition. *Assume the residue A_p is nilpotent and let $\Phi_p(z)$ be the canonical fundamental solution of $\widehat{\nabla}$ near $p \in \mathcal{P}$. Then*

$$(z - p)^{-A_p} \cdot \Phi_p(z) \rightarrow 1 \text{ as } z \rightarrow p.$$

PROOF. Write

$$(z - p)^{-A_p} H_p(z) (z - p)^{A_p} = H_p(z) + [(z - p)^{-A_p}, H_p(z)] (z - p)^{A_p}.$$

By definition the first term tends to 1 as $z \rightarrow p$. Writing $H_p(z) = 1 + (z - p)J_p(z)$ with J_p holomorphic at $z = p$, the second term can be rewritten as $(z - p)[(z - p)^{-A_p}, J_p(z)]$. Since A_p is nilpotent and $(z - p)$ term kills all powers of $\log(z - p)$, this tends to zero as $z \rightarrow p$. \square

7.4. **Regularised parallel transport.** Suppose now that $\gamma: [0, 1] \rightarrow \mathbb{C}$ is a path such that $\gamma(0, 1) \subset \mathbb{C} \setminus \mathcal{P}$ but which starts at a pole $p \in \mathcal{P}$ and ends at a pole $q \in \mathcal{P}$. For $0 < s < t < 1$ let $\gamma_{[s, t]}$ denote the path in $\mathbb{C} \setminus \mathcal{P}$ obtained by restricting γ to the interval $[s, t]$. It follows from Proposition 7.3 that the limit

$$\text{PT}_\gamma^{\text{reg}} = \lim_{\substack{s \rightarrow 0 \\ t \rightarrow 1}} \left[(\gamma(t) - q)^{-A_q} \cdot \text{PT}_{\gamma_{[s, t]}} \cdot (\gamma(s) - p)^{A_p} \right] \quad (37)$$

is well-defined. Its value is called the *regularized parallel transport* of $\widehat{\nabla}$ along γ . Such limits will be important in our computations of Stokes factors.

Lemma. *If $p_1 \neq q$ and $p_n \neq p$ then the integral*

$$I_{\gamma,n}(p_1, \dots, p_n) = \lim_{\substack{s \rightarrow 0 \\ t \rightarrow 1}} \int_{\gamma_{[s,t]}}^* \frac{dz}{z - p_1} \circ \dots \circ \frac{dz}{z - p_n}$$

is convergent.

PROOF. This is proved in [9, Section 2.9]. \square

Proposition. *Assume the residues of $\widehat{\nabla}$ are nilpotent. Suppose that $P: U \rightarrow V$ and $Q: V \rightarrow W$ are linear maps such that $A_p \cdot P = 0$ and $Q \cdot A_q = 0$. Then,*

$$Q \cdot \text{PT}_{\gamma}^{\text{reg}} \cdot P = \lim_{\substack{s \rightarrow 0 \\ t \rightarrow 1}} Q \cdot \text{PT}_{\gamma_{[s,t]}} \cdot P,$$

and there is a series expansion

$$Q \cdot \text{PT}_{\gamma}^{\text{reg}} \cdot P = Q \left(1 + \sum_{n \geq 1} \sum_{\substack{p_1, \dots, p_n \in \mathcal{P} \\ p_1 \neq q, p_n \neq p}} I_{\gamma,n}(p_1, \dots, p_n) \cdot A_{p_1} \cdots A_{p_n} \right) P$$

which is absolutely convergent in n .

PROOF. The first statement is clear because $(\gamma(s) - p)^{A_p} \cdot P = P$ and $Q \cdot (\gamma(t) - q)^{-A_q} = Q$. To obtain the series expansion consider first fixing the residues A_p and then rescaling them by an element $\lambda \in \mathbb{C}$. For each $0 < s < t < 1$, the function

$$Q \cdot \text{PT}_{\gamma_{[s,t]}} \cdot P$$

is then an analytic function of λ . Theorem 7.2 shows that it has Taylor series

$$Q \cdot P + \sum_{n \geq 1} \sum_{\substack{p_1, \dots, p_n \in \mathcal{P} \\ p_1 \neq q, p_n \neq p}} \left(I_{\gamma_{[s,t],n}(p_1, \dots, p_n) Q \cdot A_{p_1} \cdots A_{p_n} \cdot P \right) \lambda^n.$$

since the terms such that $p_1 = q$ or $p_n = p$ are killed by Q and P respectively.

The following standard result of complex analysis completes the proof. Let f_{ϵ} be holomorphic functions on \mathbb{C} defined for $\epsilon \in (0, 1)$. Suppose that on some closed disc $f_{\epsilon} \rightarrow f$ uniformly as $\epsilon \rightarrow 0$. Then f is holomorphic on the interior of the disc, and has a Taylor expansion there whose coefficients are the limits of the Taylor coefficients of the f_{ϵ} . \square

7.5. For later use we need one more result on regularized parallel transport maps. Let α be a path starting at a point p (possibly a pole of $\widehat{\nabla}$) and ending at a pole q , and β a path starting from the pole q , both paths otherwise avoiding the poles of $\widehat{\nabla}$. The concatenation $\beta \cdot \alpha$ can be deformed in two ways to give two paths γ_+ , γ_- which avoid the point q by a small anticlockwise (resp. clockwise) half circle.

Proposition. *Assume that the residues of $\widehat{\nabla}$ are nilpotent. Then*

$$\text{PT}_{\gamma_+}^{\text{reg}} - \text{PT}_{\gamma_-}^{\text{reg}} = \text{PT}_{\beta}^{\text{reg}} \cdot (e^{2\pi i A_q} - 1) \cdot \text{PT}_{\alpha}^{\text{reg}}.$$

PROOF. Deforming the paths slightly we can assume that $\alpha(1 - \epsilon) = \beta(\epsilon)$ for small enough $\epsilon > 0$. Consider the expression

$$\text{PT}_{\beta_{[s,1]}} \cdot (\beta(s) - q)^{A_q} \cdot (e^{2\pi i A_q} - 1) \cdot (\alpha(t) - q)^{-A_q} \cdot \text{PT}_{\alpha_{[u,t]}} \cdot (\alpha(u) - p)^{A_p}.$$

Its limit as $s, u \rightarrow 0$ and $t \rightarrow 1$ is the right hand side of the stated identity. Take $s = u = \epsilon$ and $t = 1 - \epsilon$. Since z^{A_q} commutes with e^{A_q} , the expression can be rewritten as

$$\text{PT}_{\beta_{[\epsilon,1]}} \cdot (e^{2\pi i A_q} - 1) \cdot \text{PT}_{\alpha_{[\epsilon,1-\epsilon]}} \cdot (\alpha(\epsilon) - p)^{A_p}.$$

Let δ be a small loop around q starting at a point z . Parallel transport of the canonical fundamental solution shows that

$$\text{PT}_{\delta} = H_q(z) e^{2\pi i A_q} H_q^{-1}(z).$$

Thus the left hand side of the stated identity is

$$\text{PT}_{\beta_{[\epsilon,1]}} \cdot H_q(\beta(\epsilon)) \cdot (e^{2\pi i A_q} - 1) \cdot H_q^{-1}(\beta(\epsilon)) \cdot \text{PT}_{\alpha_{[\epsilon,1-\epsilon]}} \cdot (\alpha(\epsilon) - p)^{A_p}.$$

As $\epsilon \rightarrow 0$, $\beta(\epsilon) \rightarrow q$ and $H_q(\beta(\epsilon)) \rightarrow 1$. This gives the result. \square

7.6. Let $(z_1, \dots, z_n) \in \mathbb{C}^n$, $z \in \mathbb{C}$ and assume that $z = z_i$ for a unique i . Assume that α is a path ending at z and β is a path starting at z , both paths otherwise avoiding the points z_i . Let γ_+ (resp. γ_-) be the paths obtained by deforming the concatenation $\beta \cdot \alpha$ by avoiding the point z_i by a small anticlockwise (resp. clockwise) half circle.

Corollary. *The following holds*

$$\begin{aligned} I_{\gamma_+,n}(z_1, \dots, z_n) - I_{\gamma_-,n}(z_1, \dots, z_n) \\ = 2\pi i \cdot I_{\alpha, i-1}(z_1, \dots, z_{i-1}) \cdot I_{\beta, n-i}(z_{i+1}, \dots, z_n). \end{aligned}$$

PROOF. This is proved in [9, Cor. 2.6]. Alternatively it can easily be obtained by equating coefficients in both sides of Proposition 7.5. \square

8. FOURIER-LAPLACE TRANSFORM

In this section, we prove Theorem 2.5 for the general linear groups by using the Fourier-Laplace transform of the connection ∇ .

8.1. Let V be a complex, finite-dimensional vector space, \mathcal{V} the holomorphically trivial vector bundle on \mathbb{P}^1 with fibre V and ∇^V the meromorphic connection on \mathcal{V} given by

$$\nabla^V = d - \left(\frac{Z}{t^2} + \frac{F}{t} \right) dt \quad (38)$$

where $Z, F \in \mathfrak{gl}(V)$. The assumptions (Z)-(f) of Section 2.2 translate into the following ones:

- (Z) Z is diagonalisable. We denote the roots of the minimal polynomial of Z by z_1, \dots, z_m , the corresponding eigenspaces by V_1, \dots, V_m and let P_1, \dots, P_m be the projections corresponding to the decomposition

$$V = V_1 \oplus \dots \oplus V_m \quad (39)$$

- (F) The diagonal blocks of F with respect to the decomposition (39) are zero, that is $P_i F P_i = 0$ for any i .

According to Definition 2.4, the Stokes rays of ∇^V are the rays $\mathbb{R}_{>0} \cdot (z_i - z_j)$, $1 \leq i \neq j \leq m$.

8.2. Let \widehat{V} be another copy of the trivial vector bundle on \mathbb{P}^1 with fibre V and consider the Fuchsian connection $\widehat{\nabla}^V$ on \widehat{V} with poles at the points z_1, \dots, z_m given by

$$\widehat{\nabla}^V = d - \sum_{i=1}^m \frac{P_i F}{z - z_i} dz$$

The connection $\widehat{\nabla}^V$ is of the form considered in Section 7. Moreover, since

$$(P_i F)^2 = (P_i F P_i) F = 0$$

by assumption (F), the residues $A_i = P_i F$ are nilpotent. In particular, $\widehat{\nabla}^V$ is non-resonant.

8.3. Fix a pole z_i and let $Q_i : V_i \hookrightarrow V$ be the inclusion. Let U_i be a connected and simply-connected neighborhood of z_i in $\mathbb{P}^1 \setminus \{z_1, \dots, \widehat{z_i}, \dots, z_m\}$.

Lemma. *There is a unique horizontal section $\phi^{(z_i)}$ of $\widehat{\nabla}^V$ defined on U_i and taking values in $\text{Hom}_{\mathbb{C}}(V_i, V)$ which is regular at z_i and such that $\phi^{(z_i)}(z_i)$ is the inclusion $Q_i : V_i \hookrightarrow V$.*

PROOF. Let $\Phi_i : U_i \rightarrow GL(V)$ be the canonical fundamental solution of $\widehat{\nabla}^V$ at z_i (see §7.3). Since $P_i F Q_i = 0$ by assumption (F),

$$\phi^{(z_i)}(z) := \Phi_i(z) \cdot Q_i = H_i(z)(z - z_i)^{P_i F} \cdot Q_i = H_i(z) \cdot Q_i$$

gives the required solution. Uniqueness is straightforward. \square

8.4. Fix a pole z_i , an admissible ray $r = \mathbb{R}_{>0} \cdot e^{i\varphi}$, and set

$$Y_r^{(z_i)}(t) = \frac{1}{t} \int_{z_i+r} \phi^{(z_i)}(z) e^{-z/t} dz \quad (40)$$

Proposition.

- (i) The integral (40) is convergent for any t in the half-plane \mathbb{H}_r .
- (ii) The corresponding function $Y_r^{(z_i)} : \mathbb{H}_r \rightarrow \text{Hom}_{\mathbb{C}}(V_i, V)$ is holomorphic and satisfies

$$\frac{dY_r^{(z_i)}}{dt} = \left(\frac{Z}{t^2} + \frac{F}{t} \right) Y_r^{(z_i)} \quad (41)$$

- (iii) $Y_r^{(z_i)} \cdot e^{z_i/t}$ tends to the inclusion $Q_i : V_i \hookrightarrow V$ as $t \rightarrow 0$ in \mathbb{H}_r .

PROOF. We drop the superscript (z_i) and the subscripts r and $z_i + r$ from the notation and use primes for derivatives. (i) follows from the fact that since $\widehat{\nabla}^V$ has regular singularities, $\phi^{(z_i)}$ grows at most polynomially as $z \rightarrow \infty$, while

$$|e^{-z/t}| = |e^{-z_i/t}| e^{-\frac{|z-z_i|}{|t|} \cos(\varphi - \arg(t))}$$

decreases exponentially as $z \rightarrow \infty$ along $z_i + \mathbb{R}_{>0} \cdot e^{i\varphi}$, provided $\arg(t) \in (\varphi - \frac{\pi}{2}, \varphi + \frac{\pi}{2})$.

(ii) Differentiating the defining integral for $Y(t)$ gives

$$Y'(t) = -\frac{1}{t}Y(t) + \frac{1}{t^3} \int \phi(z) z e^{-z/t} dz.$$

Integrating the second term by parts gives

$$Y'(t) = -\frac{1}{t}Y(t) + \frac{1}{t^2} Q_i z_i e^{-z_i/t} + \frac{1}{t^2} \int \frac{d}{dz} (z\phi(z)) e^{-z/t} dz.$$

Expanding the derivative, two terms cancel, giving

$$Y'(t) = \frac{1}{t^2} Q_i z_i e^{-z_i/t} + \frac{1}{t^2} \int \left(\sum_j \frac{P_j F}{z - z_j} \phi(z) \right) z e^{-z/t} dz.$$

Taking the finite sum outside the integral and writing $z = (z - z_j) + z_j$ gives a sum of two terms. The first is

$$\frac{1}{t^2} \sum_j P_j F \int \phi(z) e^{-z/t} dz = \frac{1}{t} \left(\sum_j P_j \right) F Y(t) = \frac{F}{t} Y(t).$$

The second is

$$\frac{1}{t^2} \left(Q_i z_i e^{-z_i/t} + \sum_j z_j P_j F \int \frac{\phi(z) e^{-z/t}}{z - z_j} dz \right). \quad (42)$$

To simplify this expression, we integrate by parts the defining expression for $Y(t)$ to get

$$\begin{aligned} Y(t) &= Q_i e^{-z_i/t} + \int \phi'(z) e^{-z/t} dz \\ &= Q_i e^{-z_i/t} + \sum_j P_j F \int \frac{\phi(z) e^{-z/t}}{z - z_j} dz \end{aligned}$$

which, upon being multiplied by P_j yields

$$P_j Y(t) = \delta_{ij} Q_i e^{-z_i/t} + P_j F \int \frac{\phi(z) e^{-z/t}}{z - z_j} dz$$

Substituting this into (42) shows that the latter is equal to

$$\frac{1}{t^2} \sum_j z_j P_j Y(t) = \frac{Z}{t^2} Y(t)$$

and therefore that Y satisfies (41) as claimed.

(iii) The limiting behaviour of $Y(t)$ as $t \rightarrow 0$ in \mathbb{H}_r follows at once from the fact that Y has an asymptotic expansion in \mathbb{H}_r with constant term 1. This in turn is a consequence of the Lemma below. \square

Lemma (Watson [11]). *Set*

$$Y(t) = \int_0^\infty \phi(z) e^{-z/t} dz,$$

and suppose that $\phi(z)$ is analytic at $z = 0$ with Taylor series

$$\phi(z) = \sum_{n \geq 0} \frac{a_n}{n!} z^n.$$

Then $Y(t)$ has the asymptotic expansion

$$Y(t) \sim \sum_{n=0}^{\infty} a_n t^{n+1}.$$

as $t \rightarrow 0$ in the half-plane $\operatorname{Re}(t) > 0$. \square

8.5. Proof of Theorem 2.5 for $\mathbf{G} = \mathbf{GL}(V)$. Let the functions $Y_r^{(z_i)}: \mathbb{H}_r \rightarrow \operatorname{Hom}(V_i, V)$ be given by Proposition 8.4 and define a map $Y_r^V: \mathbb{H}_r \rightarrow \operatorname{End}(V)$ by

$$Y_r^V = \sum_i Y_r^{(z_i)} P_i$$

By Proposition 8.4, $Y_r^V(t)$ is a fundamental solution of ∇^V and $Y_r^V(t) \cdot e^{Z/t}$ tends to the identity as $t \rightarrow 0$ in \mathbb{H}_r .

To prove uniqueness, let $Y_1, Y_2: \mathbb{H}_r \rightarrow GL(V)$ be two holomorphic functions satisfying (7)–(8). Then $g = Y_2^{-1} \cdot Y_1$ is a locally constant and therefore constant $GL(V)$ -valued function on \mathbb{H}_r such that $e^{-Z/t} \cdot g \cdot e^{Z/t}$ tends to 1 as $t \rightarrow 0$ in \mathbb{H}_r . Applying Proposition 6.3, we see that $g = \exp(X)$ where $X \in \mathfrak{gl}(V)$ only has components along the $\operatorname{ad}(Z)$ -eigenspaces corresponding to eigenvalues $\lambda \in r$. Since r is an admissible ray of ∇^V , r contains no such eigenvalues and $X = 0$. \square

9. COMPUTATION OF THE STOKES FACTORS FOR $GL(V)$

We now explicitly compute the Stokes factors of the connection ∇^V in terms of multilogarithms. Retain the notation of Section 8.

9.1. Fix a pole z_i and let ℓ be a Stokes ray of ∇^V , so that $z_i + \ell$ contains some of the poles of $\widehat{\nabla}^V$. List these in order of increasing distance with respect to z_i as z_{i_j} , $j = 1, \dots, p$ and set $z_{i_0} = z_i$. Let r_\pm be small anticlockwise and clockwise rotations of ℓ respectively such that the closed, convex sectors $\overline{\Sigma}_\pm$ determined by $z_i + r_\pm$ and $z_i + \ell$ only contain the poles z_{i_j} , $j = 0, \dots, p$.

Proposition. *The following holds on $\mathbb{H}_{r_+} \cap \mathbb{H}_{r_-}$*

$$Y_{r_-}^{(z_i)} = Y_{r_+}^{(z_i)} + 2\pi i \sum_{j=1}^p Y_{r_+}^{(z_{i_j})} \circ P_{i_j} F \circ \operatorname{PT}_{C_{i_j i}}^{\operatorname{reg}} \circ Q_i \quad (43)$$

where $C_{i,j}$ is a small perturbation of the oriented line segment $[z_i, z_{i_j}]$ which avoids the poles z_{i_k} , with $1 \leq k \leq j-1$ by using anticlockwise arcs of circle around them.

PROOF. For any $0 \leq j \leq p$, let γ_j be a small perturbation of the ray $z_i + \ell$ which avoids the poles z_{i_k} , with $1 \leq k \leq j$, by going into $\overline{\Sigma}_-$ and the poles z_{i_k} , with $j+1 \leq k \leq p$ by going into $\overline{\Sigma}_+$. By Cauchy's theorem, and the fact that $\phi^{(z_i)}(z)e^{-z/t}$ decays exponentially as z goes to infinity along ℓ ,

$$Y_{r_+}^{(z_i)}(t) = \frac{1}{t} \int_{\gamma_0} \phi^{(z_i)}(z) e^{-z/t} dz \quad \text{and} \quad Y_{r_-}^{(z_i)}(t) = \frac{1}{t} \int_{\gamma_p} \phi^{(z_i)}(z) e^{-z/t} dz$$

We wish to compute the difference $\int_{\gamma_j} \phi^{(z_i)}(z) e^{-z/t} dz - \int_{\gamma_{j-1}} \phi^{(z_i)}(z) e^{-z/t} dz$ for any $p \geq j \geq 1$. Let $\epsilon > 0$ be small enough and write

$$\gamma_j = \gamma_{j,\epsilon}^+ \circ D_{j,\epsilon}^- \circ \gamma_{j,\epsilon}^- \quad \text{and} \quad \gamma_{j-1} = \gamma_{j,\epsilon}^+ \circ D_{j,\epsilon}^+ \circ \gamma_{j,\epsilon}^-$$

where:

- $\gamma_{j,\epsilon}^-$ is the perturbation of the straight line segment from z_i to $z_{i_j} - \epsilon e^{i\varphi}$, where $\ell = \mathbb{R}_{>0} \cdot e^{i\varphi}$, which avoids the poles z_{i_k} , with $1 \leq k \leq j-1$, by going into $\overline{\Sigma}_-$.
- $D_{j,\epsilon}^\pm$ are the arcs of circle of radius ϵ centred at z_{i_j} joining $z_{i_j} - \epsilon e^{i\varphi}$ to $z_{i_j} + \epsilon e^{i\varphi}$ in $\overline{\Sigma}_\pm$.
- $\gamma_{j,\epsilon}^+$ is the perturbation of the line $z_{i_j} + \mathbb{R}_{>\epsilon} \cdot e^{i\varphi}$ which avoids the poles z_{i_k} , with $j+1 \leq k \leq p$, by going into $\overline{\Sigma}_+$.

This yields

$$\begin{aligned} & \int_{\gamma_j} \phi^{(z_i)}(z) e^{-z/t} dz - \int_{\gamma_{j-1}} \phi^{(z_i)}(z) e^{-z/t} dz \\ &= \int_{D_{j,\epsilon}^-} \phi^{(z_i)}(z) e^{-z/t} dz - \int_{D_{j,\epsilon}^+} \phi^{(z_i)}(z) e^{-z/t} dz + \int_{\gamma_{j,\epsilon}^+} \left(\phi_-^{(z_i)} - \phi_+^{(z_i)} \right) (z) e^{-z/t} dz \end{aligned}$$

where $\phi_{\mp}^{(z_i)}(z)$ is the analytic continuation of $\phi^{(z_i)}$ along the path

$$C_{j,\epsilon}^\mp(z) = \gamma_{j,\epsilon}^+(z) \circ D_{j,\epsilon}^\mp \circ \gamma_{j,\epsilon}^-$$

and $\gamma_{j,\epsilon}^+(z)$ is the portion of $\gamma_{j,\epsilon}^+$ joining $z_{i_j} + \epsilon e^{i\varphi}$ to z . Since $C_{j,\epsilon}^\mp(z)$ differ by a small loop around z_{i_j} , Proposition 7.5 yields

$$\int_{\gamma_{j,\epsilon}^+} \left(\phi_-^{(z_i)} - \phi_+^{(z_i)} \right) (z) e^{-z/t} dz = 2\pi i \int_{\gamma_{j,\epsilon}^+} \phi^{(z_{i_j})} e^{-z/t} dz \circ P_{i_j} F \circ \text{PT}_{C_{i_j,i}} \circ Q_i$$

By Cauchy's theorem again,

$$\lim_{\epsilon \rightarrow 0} \int_{\gamma_{j,\epsilon}^+} \phi^{(z_{i_j})} e^{-z/t} dz = \int_{z_{i_j} + r_+} \phi^{(z_{i_j})} e^{-z/t} dz = t Y_{r_+}^{(z_{i_j})}(t)$$

To conclude, it suffices to show that the integrals $\int_{D_{j,\epsilon}^\pm} \phi^{(z_i)}(z) e^{-z/t} dz$ tend to zero as $\epsilon \rightarrow 0$. Let $\Phi_{i_j}(z) = H_{i_j}(z)(z - z_{i_j})^{P_{i_j} F}$ be the canonical fundamental solution of

$\widehat{\nabla}^V$ at z_{i_j} , so that $\phi^{(z_i)} = \Phi_{i_j} \cdot C_i$ for some $C_i \in \text{Hom}_{\mathbb{C}}(V_i, V)$. Then, if $\|\cdot\|$ is an algebra norm on $\text{End}(V)$,

$$\begin{aligned} \left\| \int_{D_{j,\epsilon}^{\pm}} \phi^{(z_i)}(z) e^{-z/t} dz \right\| &= \left\| \int_{D_{j,\epsilon}^{\pm}} H_{i_j}(z) (z - z_{i_j})^{P_{i_j} F} C_i e^{-z/t} dz \right\| \\ &\leq \pi \epsilon (1 + (|\ln \epsilon| + \pi) \|P_{i_j} F\|) M \end{aligned}$$

where $M = \|C_i\| \cdot \max_{|z - z_{i_j}| \leq \epsilon} \|H_{i_j}(z)\| e^{-z/t} < \infty$. \square

9.2. Let ℓ be the Stokes ray of ∇^V and $S_{\ell}^V \in \text{GL}(V)$ the corresponding Stokes factor, so that, on $\mathbb{H}_{r_-} \cap \mathbb{H}_{r_+}$

$$Y_{r_-}^V = Y_{r_+}^V \cdot S_{\ell}^V \quad (44)$$

where r_{\pm} are small anticlockwise and clockwise perturbations of ℓ . Define a partial order on the set of poles of ∇^V by

$$z_j >_{\ell} z_i \quad \text{if} \quad z_j \in z_i + \ell \quad (45)$$

The following result gives a formula for the blocks of S_{ℓ}^V corresponding to the decomposition (39) of V into eigenspaces of Z .

Theorem.

$$P_j \circ S_{\ell}^V \circ P_i = \begin{cases} 0 & \text{if } z_j \not>_{\ell} z_i \\ P_i & \text{if } z_j = z_i \\ 2\pi i \cdot P_j F \circ \text{PT}_{C_{ji}}^{\text{reg}} \circ P_i & \text{if } z_j >_{\ell} z_i \end{cases} \quad (46)$$

where C_{ji} is a small perturbation of the oriented line segment $[z_i, z_j]$ which avoids the poles $z_k \in (z_i, z_j)$ by using anticlockwise arcs of circle around them.

Thus, for $z_j >_{\ell} z_i$,

$$P_j \circ S_{\ell}^V \circ P_i = 2\pi i \cdot P_j F \left(1 + \sum_{n \geq 1} \sum_{\substack{k_1, \dots, k_n: \\ k_1 \neq j, k_n \neq i}} \int_{C_{ji}}^* \frac{dz}{z - z_{k_1}} \circ \dots \circ \frac{dz}{z - z_{k_n}} P_{k_1} F \dots P_{k_n} F \right) P_i$$

PROOF. The first statement follows from (44), the fact that $Y_{r_{\pm}}^V \cdot P_i = Y_{r_{\pm}}^{(z_i)} \cdot P_i$ and Proposition 9.1. The second from Proposition 7.4. \square

Remark. Theorem 9.2 and its proof extend the computation of Balser–Jurkat–Lutz to the case when Z has repeated eigenvalues [2].

10. TANNAKA DUALITY

In this section, we prove Theorems 2.5 and 4.5 for an arbitrary algebraic group G by relying on the results of Sections 8 and 9 and using Tannaka duality.

10.1. Let $\rho : G \rightarrow \mathrm{GL}(V)$ be a finite-dimensional representation of G and $\mathcal{V} = P \times_G V$ the holomorphically trivial vector bundle over \mathbb{P}^1 with fibre V . The connection (3) induces a meromorphic connection ∇^V on \mathcal{V} given by

$$\nabla^V = d - \left(\frac{\rho(Z)}{t^2} + \frac{\rho(f)}{t} \right) dt \quad (47)$$

Lemma. *The connection ∇^V satisfies the assumptions of Section 8.1, that is*

- $\rho(Z) \in \mathfrak{gl}(V)$ is semisimple.
- The diagonal blocks of $\rho(f)$ with respect to the eigenspace decomposition of V under $\rho(Z)$ is zero.

PROOF. This holds because $\mathfrak{g} = \mathfrak{g}^Z \oplus [Z, \mathfrak{g}]$ and $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ is equivariant with respect to the adjoint action of Z . \square

Remark. The Stokes rays of ∇ and ∇^V are related, though not in an entirely straightforward way. If ℓ is a Stokes ray of ∇ , and the restriction of ρ to the subalgebra $\bigoplus_{\zeta \in \ell} \mathfrak{g}_\zeta \subset \mathfrak{g}$ is not zero, then ℓ is also a Stokes ray of ∇^V . This is the case if ρ is faithful as a representation of \mathfrak{g} for example, and therefore if G is semisimple. In general however, simple examples show that a Stokes ray of ∇ need not be one of ∇^V and that, conversely, a Stokes ray of ∇^V need not be one of ∇ .

10.2. Let $\mathrm{Rep}(G)$ be the category of finite-dimensional representations of G . We establish below the naturality of the canonical fundamental solutions of the connections ∇^V with respect to tensor products and homomorphisms in $\mathrm{Rep}(G)$.

The union \mathcal{L} of the sets of Stokes rays of the connections ∇^V , $V \in \mathrm{Rep}(G)$ is at most countable. Fix $r \notin \mathcal{L}$ and let $\mathbb{H}_r \subset \mathbb{C}^*$ be the corresponding half-plane (6).

Proposition. *Let $\{Y_r^V\}_{V \in \mathrm{Rep}(G)}$ be a family of holomorphic functions $Y_r^V : \mathbb{H}_r \rightarrow \mathrm{GL}(V)$ such that*

$$\frac{dY_r^V}{dt} = \left(\frac{\rho(Z)}{t^2} + \frac{\rho(f)}{t} \right) Y_r^V$$

$$Y_r^V \cdot e^{\rho(Z)/t} \rightarrow 1 \quad \text{as } t \rightarrow 0 \text{ in } \mathbb{H}_r$$

Then, the following holds for any $V_1, V_2 \in \mathrm{Rep}(G)$ and $T \in \mathrm{Hom}_G(V_1, V_2)$,

$$T Y_r^{V_1} = Y_r^{V_2} T \quad (48)$$

$$Y_r^{V_1 \otimes V_2} = Y_r^{V_1} \otimes Y_r^{V_2} \quad (49)$$

PROOF. (48) follows from the uniqueness part of Theorem 2.5 for the group $\mathrm{GL}(V_1 \otimes V_2)$ (see §8.5) since both sides are fundamental solutions of $\nabla^{V_1 \otimes V_2} = \nabla^{V_1} \otimes 1 + 1 \otimes \nabla^{V_2}$ having the required asymptotic properties on \mathbb{H}_r . (49) follows in a similar manner. Namely, consider the element

$$C = (Y_r^{V_2})^{-1} \cdot T Y_r^{V_1} \in \mathrm{Hom}_{\mathbb{C}}(V_1, V_2)$$

Condition (10.2) implies that $e^{-\rho_{V_2}(Z)/t} \cdot (C - T) \cdot e^{\rho_{V_1}(Z)/t}$ tends to 0 as $t \rightarrow 0$ in \mathbb{H}_r . Applying Lemma 6.2 to $u = C - T \in \mathrm{Hom}_{\mathbb{C}}(V_1, V_2) = U$, we see that the

only non-trivial components of $C - T$ along the eigenspace decomposition of the G -module $\text{Hom}_{\mathbb{C}}(V_1, V_2)$ under Z correspond to eigenvalues lying in r . Since r is not a Stokes ray of $\nabla^{\text{End}(V_1, V_2)}$ however it follows that $C - T = 0$. \square

10.3. Proof of Theorem 2.5. Assume first that the ray r is admissible for all connections ∇^V , $V \in \text{Rep}(G)$, that is that $r \notin \mathcal{L}$. For any $V \in \text{Rep}(G)$, let $Y_r^V : \mathbb{H}_r \rightarrow \text{GL}(V)$ be the corresponding canonical fundamental solution of ∇^V . By (48), the collection $\{Y_r^V(t)\}$ defines a function Y_r on \mathbb{H}_r with values in $\widehat{U\mathfrak{g}}$. By (49), Y_r takes values in $G \subset \widehat{U\mathfrak{g}}$ since, by Tannaka duality G is the set of grouplike elements of $\widehat{U\mathfrak{g}}$ [19]. Since Y_r^V satisfies the properties (7)–(8) for any representation V of G , Y_r^Y is a holomorphic G -valued function which satisfies these same properties.

Assume now that r is admissible for ∇ and let r_+, r_- be small anticlockwise and clockwise perturbations of r such that $r_{\pm} \notin \mathcal{L}$ and the closed convex sector $\overline{\Sigma} \subset \mathbb{C}^*$ bounded by r_{\pm} does not contain any Stokes ray of ∇ . The element $g \in G$ defined by $Y_{r_-} = Y_{r_+} g$ on $\mathbb{H}_{r_-} \cap \mathbb{H}_{r_+}$ is such that $e^{-Z/t} g e^{Z/t} \rightarrow 1$ as $t \rightarrow 0$ in $\mathbb{H}_{r_-} \cap \mathbb{H}_{r_+}$. By Proposition 2.6, $g = \exp(X)$ where X lies in the span of the eigenspaces of $\text{ad}(Z)$ corresponding to eigenvalues contained in $\overline{\Sigma}$. Since there are none, $X = 0$ so that $Y_{r_{\pm}}$ patch to a fundamental solution of ∇ having the required asymptotic property on $\mathbb{H}_{r_-} \cup \mathbb{H}_{r_+} \supset \mathbb{H}_r$. \square

10.4. Proof of Theorem 4.5. Let $\rho : G \rightarrow \text{GL}(V)$ be a finite-dimensional representation of G . We begin by reworking the formula for the Stokes factors of the linear connection ∇^V obtained in Theorem 9.2.

Let z_1, \dots, z_m be the roots of the minimal polynomial of $\rho(Z)$ and $P_1, \dots, P_m \in \text{End}(V)$ the corresponding eigenprojections. Let ℓ be a Stokes ray of ∇^V and z_i, z_j two eigenvalues of $\rho(Z)$ such that $z_j \in z_i + \ell$. Since $f = \sum_{\alpha \in \Phi^Z} f_{\alpha}$, the product of operators $P_j f P_{k_1} f \cdots P_{k_n} f P_i$ appearing in Theorem 9.2 is equal to

$$\sum_{\alpha_0, \dots, \alpha_n \in \Phi^Z} P_j f_{\alpha_0} P_{k_1} f_{\alpha_1} \cdots P_{k_n} f_{\alpha_n} P_i$$

where we abusively denote $\rho(f)$ by f .

Given that $P_{k_m} f_{\alpha_m} P_{k_{m+1}} = 0$ unless $z_{k_m} = z_{k_{m+1}} + Z(\alpha_m)$, the sum over k_1, \dots, k_n becomes one over the roots $\alpha_0, \dots, \alpha_n$ with $z_{k_m} = z_j - Z(\alpha_0 + \cdots + \alpha_{m-1})$ and the constraint $Z(\alpha_0 + \cdots + \alpha_m) = z_j - z_i$. It follows that $(2\pi i)^{-1} P_j S_{\ell}^V P_i$ is equal to

$$P_j f P_i + \sum_{\substack{\alpha_0, \dots, \alpha_n \in \Phi^Z: \\ Z(\alpha_0 + \cdots + \alpha_n) = z_j - z_i}} \int_{C_{j_i}}^* \frac{dz}{z - z_j + s_0} \circ \cdots \circ \frac{dz}{z - z_j + s_{n-1}} \cdot P_j f_{\alpha_0} f_{\alpha_1} \cdots f_{\alpha_n} P_i$$

where $s_m = Z(\alpha_0 + \cdots + \alpha_m)$, so that $s_n = z_j - z_i$. By Lemma 7.1, the change of variable $z \rightarrow z_j - z$ in the iterated integral yields

$$\int_{C_{0, s_n}}^* \frac{dz}{z - s_0} \circ \cdots \circ \frac{dz}{z - s_{n-1}} = \int_{\overline{C_{0, s_n}}} \frac{dz}{z - s_0} \circ \cdots \circ \frac{dz}{z - s_{n-1}}$$

where C_{0, s_n} is a small perturbation of $[s_n, 0]$ which avoids the poles s_k , $1 \leq k \leq n-1$ such that $s_k \in (s_n, 0)$ by using anticlockwise arcs of circles around them, $\overline{C_{0, s_n}}(t) =$

$C_{0,s_n}(1-t)$ is C_{0,s_n} with the opposite orientation, and the above equality follows from (i) of Lemma 7.1 and (35). Since $\overline{C_{0,s_n}}$ is a perturbation of $[0, s_n]$ which avoids the s_k such that $s_k \in (s_n, 0)$ by *clockwise* arcs of circles around them, it follows that

$$P_j S_\ell^V P_i = 2\pi i \sum_{n \geq 1} \sum_{\substack{\alpha_1, \dots, \alpha_n \in \Phi^Z: \\ Z(\alpha_1 + \dots + \alpha_n) = z_j - z_i}} M_n(Z(\alpha_1), \dots, Z(\alpha_n)) P_j f_{\alpha_1} \cdots f_{\alpha_n} P_i$$

where the M_n are the functions defined in Section 4.5. Since S_ℓ^V is equal to $1 + \sum_{i,j: z_j - z_i \in \ell} P_j S_\ell^V P_i$, this yields

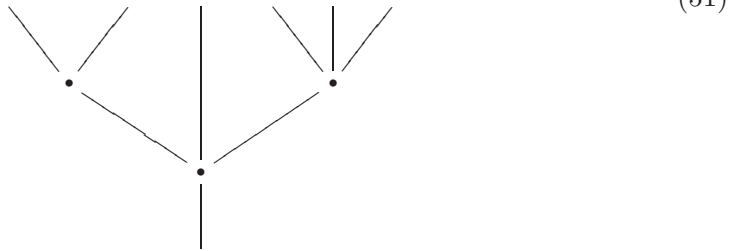
$$S_\ell^V = 1 + 2\pi i \sum_{n \geq 1} \sum_{\substack{\alpha_1, \dots, \alpha_n \in \Phi^Z: \\ Z(\alpha_1 + \dots + \alpha_n) \in \ell}} M_n(Z(\alpha_1), \dots, Z(\alpha_n)) f_{\alpha_1} \cdots f_{\alpha_n} \quad (50)$$

Let now ℓ be a Stokes ray of ∇ . We wish to show that $\rho(S_\ell)$ is given by the right-hand side of (50). If ℓ is also a Stokes ray of ∇^V , this follows from the fact that $\rho(S_\ell) = S_\ell^V$. If, on the other hand, ℓ is not a Stokes ray of ∇^V , then $\rho(S_\ell) = 1$ which is also the value returned by the right-hand side of (50). Indeed, in this case $z_j - z_i \notin \ell$ for any i, j , so that $P_j f_{\alpha_1} \cdots f_{\alpha_n} P_i = 0$ whenever $Z(\alpha_1 + \dots + \alpha_n) \in \ell$. \square

11. INVERSION OF NON-COMMUTATIVE POWER SERIES

In this section, we compute the Taylor series of the inverse of the Stokes map, thereby proving Theorem 4.8. We shall do so by working out a non-commutative analogue of the compositional inversion of a formal power series.

11.1. We shall need some notation on trees. A tree T is a finite, connected and simply-connected graph. We denote the set of edges of T by $E(T)$ and the set of vertices by $V(T)$. A plane tree is a tree T together with a cyclic ordering of the incident edges at each vertex. A tree has both internal and external edges; a rooted plane tree is a plane tree with a distinguished external edge called the root; the other external edges are then called the leaves. We draw plane trees in such a way that the cyclic ordering of the edges incident at a given vertex is the natural clockwise ordering induced by the embedding in the plane. For example



Note that the incoming edges at a vertex v have a canonical ordering. Similarly the leaves of T have a canonical ordering.

11.2. Let U and V be complex vector spaces. By a *non-commutative (NC) power series* $\phi: U \rightarrow V$ we mean a sequence of linear maps

$$\phi_n: U^{\otimes n} \rightarrow V, \quad n \geq 1.$$

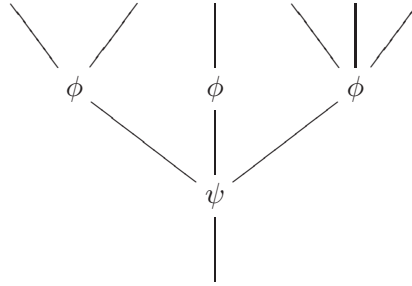
Two such power series $\phi: U \rightarrow V$ and $\psi: V \rightarrow W$ can be composed to give a power series $\psi \circ \phi: U \rightarrow W$ by using the following rule

$$(\psi \circ \phi)_n(u_1, \dots, u_n) = \sum_{k=1}^n \sum_{0=i_0 < \dots < i_k=n} \psi_k(\phi_{i_1-i_0}(u_{i_0+1}, \dots, u_{i_1}), \dots, \phi_{i_k-i_{k-1}}(u_{i_{k-1}+1}, \dots, u_{i_k})). \quad (52)$$

This sum is best visualized as a sum over plane rooted trees of height two with the tensors ϕ and ψ labelling the vertices, and the inputs u_1, \dots, u_n labelling the leaves. For example, in the sum for $(\psi \circ \phi)_6$, the term

$$\psi_3(\phi_2(u_1, u_2), \phi_1(u_3), \phi_3(u_4, u_5, u_6))$$

corresponds to the tree



The composition law (52) is easily checked to be associative. We thus obtain a category \mathcal{NC} whose objects are vector spaces and whose morphisms are NC power series. The identity morphism corresponding to a vector space V is the power series $\text{id}_V: V \rightarrow V$ given by $\text{id}_1 = \text{id}_V$ and $\text{id}_n = 0$ for $n > 1$.

Lemma. *A NC power series $\phi: U \rightarrow V$ is an isomorphism iff the linear map $\phi_1: U \rightarrow V$ is an isomorphism.*

PROOF. If ϕ_1 is an isomorphism one can inductively solve the equation $\psi \circ \phi = \text{id}$ for $\psi_n: V^{\otimes n} \rightarrow V$. Similarly one can find ψ' with $\phi \circ \psi' = \text{id}$. By general nonsense $\psi = \psi'$ is then an inverse for ϕ . The converse is obvious. \square

11.3. Suppose $\phi: V \rightarrow V$ is a NC power series with $\phi_1 = \text{id}_V$. For any rooted plane tree T with n leaves, we can form a linear map

$$\phi_T: V^{\otimes n} \rightarrow V$$

by thinking of the leaves of T as inputs and using the vertices of T to compose the tensors ϕ_k . For example either of the trees above corresponds to the map

$$\phi_T(v_1, \dots, v_6) = \phi_6(\phi_2(v_1, v_2), v_3, \phi_3(v_4, v_5, v_6)).$$

We shall only consider trees all of whose vertices have valency ≥ 3 since the assumption $\phi_1 = \text{id}_V$ implies that vertices of valency 2 do not contribute anything new.

Theorem. *Let ϕ be a NC power series satisfying $\phi_1 = \text{id}_V$. Then $\phi_1^{-1} = \text{id}_V$ and, for $n > 1$*

$$\phi_n^{-1}(v_1, \dots, v_n) = \sum_T (-1)^{|V(T)|} \phi_T(v_1, \dots, v_n)$$

where the sum is over rooted plane trees with n leaves all of whose vertices have valency ≥ 3 .

PROOF. It is enough to check that if one defines ϕ^{-1} by the given formula the composite $\phi \circ \phi^{-1}$ is the identity. Clearly $(\phi \circ \phi^{-1})_1 = \text{id}_V$. For $n > 1$, expanding the composite $(\phi \circ \phi^{-1})_n$ gives a finite sum of signed terms of the form $\phi_T(v_1, \dots, v_n)$ for trees T with n leaves and vertices of valency ≥ 3 . Each such tree appears twice: once for the term in (52) where $k = 1$, and once for a term with k equal to the valency of the root vertex of T . These two terms appear with opposite signs and hence cancel. \square

11.4. Similarly to 11.2, one can define a category \mathcal{C} whose objects are complex vector spaces and whose morphisms are *commutative power series* $\phi : U \rightarrow V$, that is sequences of linear maps $\phi_n : S^n U \rightarrow V$, $n \geq 1$ where $S^n U$ is the n th symmetric power of U . Composition is defined on tensors of the form $u^{\otimes n}$, $u \in U$, by

$$\begin{aligned} (\psi \circ \phi)_n(u, \dots, u) = \\ \sum_{k=1}^n \sum_{0=i_0 < \dots < i_k=n} \psi_k(\phi_{i_1-i_0}(u, \dots, u), \dots, \phi_{i_k-i_{k-1}}(u, \dots, u)) \end{aligned} \quad (53)$$

and then by polarisation

$$(\psi \circ \phi)_n(u_1, \dots, u_n) = \frac{1}{n!} \left. \frac{\partial}{\partial t_1} \right|_{t_1=0} \dots \left. \frac{\partial}{\partial t_n} \right|_{t_n=0} (\psi \circ \phi)_n((t_1 u_1 + \dots + t_n u_n)^{\otimes n})$$

The symmetrisation homomorphism $\sigma : S^n U \hookrightarrow U^{\otimes n}$ given by

$$\sigma(u_1 \otimes \dots \otimes u_n) = \frac{1}{n!} \sum_{\tau \in \text{Sym}_n} u_{\tau(1)} \otimes \dots \otimes u_{\tau(n)}$$

yields a surjective restriction map $\sigma^* : \text{Hom}_{\mathcal{NC}}(U, V) \rightarrow \text{Hom}_{\mathcal{C}}(U, V)$. A comparison of (52) and (53), and the fact that a linear map $S^n U \rightarrow V$ is uniquely determined by its values on tensors $u^{\otimes n}$, $u \in U$, readily shows that σ^* gives rise to a functor $\sigma^* : \mathcal{NC} \rightarrow \mathcal{C}$. In particular, Theorem 11.3 can be used to invert commutative power series.

11.5. If $U \subseteq V$ is a subspace, we shall say that a NC power series $\phi : V \rightarrow V$ preserves U if the restriction $\sigma^* \phi_n$ of each ϕ_n to $S^n U \subset V^{\otimes n}$ maps into U .

Lemma.

- (i) If ϕ, ψ preserve U , so does $\psi \circ \phi$.
- (ii) If ϕ is invertible and preserves U , then so does ϕ^{-1} .

PROOF. Denote the inclusion $U \hookrightarrow V$ by ι and let $p : V \rightarrow U$ be a projection. A NC power series Θ preserves U if, and only if $(\iota p - \text{id}) \circ \sigma^* \Theta \circ \iota = 0$. (i) now follows since

$$(\iota p - \text{id}) \circ \sigma^* (\psi \circ \phi) \circ \iota = ((\iota p - \text{id}) \circ \sigma^* \psi) \circ \iota p \circ \sigma^* \phi \circ \iota = ((\iota p - \text{id}) \circ \sigma^* \psi \circ \iota) \circ p \circ \sigma^* \phi \circ \iota = 0$$

(ii) follows similarly from

$$0 = (\iota p - \text{id}) \circ \sigma^* \text{id} \circ \iota = (\iota p - \text{id}) \circ \sigma^* \phi^{-1} \circ \sigma^* \phi \circ \iota = ((\iota p - \text{id}) \circ \sigma^* \phi^{-1} \circ \iota) \circ (p \circ \sigma^* \phi \circ \iota)$$

and the fact that the commutative power series $p \circ \sigma^* \phi \circ \iota : U \rightarrow U$ is invertible. \square

11.6. We shall be particularly interested in NC power series of the special form appearing in Section 4. These power series depend on systems of coefficients which we axiomatise as follows.

Definition. By a *transform* F we mean a sequence of functions

$$F_n : \mathbb{C}^n \rightarrow \mathbb{C}, \quad n \geq 1.$$

Given transforms F and G the composite transform $G \circ F$ is defined by the finite sum

$$(G \circ F)_n(z_1, \dots, z_n) = \sum_{k=1}^n \sum_{0=i_0 < \dots < i_k=n} \left[G_k \left(\sum_{i=i_0+1}^{i_1} z_i, \sum_{i=i_1+1}^{i_2} z_i, \dots, \sum_{i=i_{k-1}+1}^{i_k} z_i \right) \cdot \prod_{j=1}^k F_{i_j - i_{j-1}}(z_{i_{j-1}+1}, \dots, z_{i_j}) \right]. \quad (54)$$

Once again this sum is best thought of as a sum over trees of height 2. For example the term corresponding to the tree (51) is

$$G_3(z_1 + z_2, z_3, z_4 + z_5 + z_6) F_2(z_1, z_2) F_1(z_3) F_3(z_4, z_5, z_6).$$

The formula (54) defines an associative composition law on the class of transforms. The transform id with $\text{id}_1 = 1$ and $\text{id}_n = 0$ for $n > 1$ is a two-sided identity. It is easy to see that a transform F is invertible precisely if the function F_1 is nowhere vanishing. Indeed, as before, in that case one can solve the equations $G \circ F = \text{id}$ and $F \circ H = \text{id}$ inductively.

11.7. Transforms give rise to NC power series as follows. Let

$$A = \bigoplus_{\lambda \in \Lambda} A_\lambda$$

be an associative algebra over \mathbb{C} graded by a free abelian group Λ . For each $\lambda \in \Lambda$ let $\pi_\lambda: A \rightarrow A_\lambda$ be the corresponding projection map. Suppose that we are given a fixed homomorphism of abelian groups $Z: \Lambda \rightarrow \mathbb{C}$. Given a transform F the corresponding NC power series $\phi(F): A \rightarrow A$ is given by the sum

$$\phi(F)_n(a_1, \dots, a_n) = \sum_{\lambda_1, \dots, \lambda_n \in \Lambda} F_n(Z(\lambda_1), \dots, Z(\lambda_n)) \pi_{\lambda_1}(a_1) * \dots * \pi_{\lambda_n}(a_n). \quad (55)$$

It is easy to check that given transforms F and G one has

$$\phi(G \circ F) = \phi(G) \circ \phi(F),$$

and hence that the data (A, Λ, Z) defines a functor from the category of transforms to that of NC power series $A \rightarrow A$.

11.8. Suppose that F is a transform satisfying $F_1(z) = 1$ for all z . The method of Theorem 11.3 allows us to give an explicit formula for the inverse of F . We first associate a function

$$F_T: \mathbb{C}^n \rightarrow \mathbb{C}$$

to a plane rooted tree with n leaves T in the following way. Identify the leaves of T with their canonical order with the set $1, \dots, n$. For each edge $e \in E(T)$ let $I(e) \subset \{1, \dots, n\}$ be the set of vertices lying above e and define the partial sum $s_e: \mathbb{C}^n \rightarrow \mathbb{C}$

$$s_e = s_e(z_1, \dots, z_n) = \sum_{i \in I(e)} z_i.$$

To each vertex $v \in V(T)$ associate a factor

$$F_v(z_1, \dots, z_n) = F_m(s_{e_1}, s_{e_2}, \dots, s_{e_m}),$$

where $m + 1$ is the valency of v and e_0, e_1, \dots, e_m are the incident edges with their clockwise ordering, with e_0 being the outward pointing edge. Then define F_T to be the product over vertices

$$F_T(z_1, \dots, z_n) = \prod_{v \in V(T)} F_v(z_1, \dots, z_n). \quad (56)$$

For example, for the tree T depicted above

$$F_T(z_1, \dots, z_6) = F_2(z_1, z_2) F_3(z_4, z_5, z_6) F_3(z_1 + z_2, z_3, z_4 + z_5 + z_6).$$

The same argument as for Theorem 11.3 gives

Proposition. *Suppose F is a transform satisfying $F_1(z) = 1$ for all z . Then $F_1^{-1}(z) = 1$ for all z and, for $n > 1$*

$$F_n^{-1}(z_1, \dots, z_n) = \sum_T (-1)^{|V(T)|} F_T(z_1, \dots, z_n)$$

where the sum is over rooted plane trees with n leaves all of whose vertices have valency ≥ 3 .

11.9. The transforms most relevant to us have the following additional property.

Definition. A *Lie transform* is a transform F such that, for any $n \geq 1$, $z_1, \dots, z_n \in \mathbb{C}$ and non-commuting variables x_1, \dots, x_n , the finite sum

$$\sum_{\sigma \in \text{Sym}_n} F_n(z_{\sigma(1)}, \dots, z_{\sigma(n)}) x_{\sigma(1)} \cdots x_{\sigma(n)}$$

is a Lie polynomial in x_1, \dots, x_n .

Lemma.

- (i) If F and G are Lie transforms, then so is $G \circ F$.
- (ii) If F is an invertible Lie transform, then so is F^{-1} .

PROOF. We begin by giving an alternative characterisation of a Lie transform H in terms of the NC power series $\phi(H)$ introduced in 11.7.

For any $n \geq 1$, let \mathcal{L}_n be the free Lie algebra on generators x_1, \dots, x_n . Its enveloping algebra $A = U\mathcal{L}_n$ possesses a grading by $\Lambda = \mathbb{Z}^n$ given by $\deg(x_i) = e_i$, where $\{e_1, \dots, e_n\}$ is the standard basis of \mathbb{Z}^n . A transform H is a Lie transform if, and only if $\phi(H) : U\mathcal{L}_n \rightarrow U\mathcal{L}_n$ preserves $\mathcal{L}_n \subset U\mathcal{L}_n$ in the sense of §11.5 for every $n \geq 1$ and homomorphism $Z : \mathbb{Z}^n \rightarrow \mathbb{C}$.

Indeed, if H is a Lie transform, then for any $n \geq 1$, $x \in \mathcal{L}_n$ and $m \geq 1$,

$$\phi(H)_m(x^{\otimes m}) = \sum_{\lambda_1, \dots, \lambda_m} H_m(Z(\lambda_1), \dots, Z(\lambda_m)) \pi_{\lambda_1}(x) \cdots \pi_{\lambda_m}(x)$$

is a Lie polynomial in the variables $\pi_{\lambda}(x)$ and therefore lies in \mathcal{L}_n . Conversely, if $\phi(H)$ preserves \mathcal{L}_n , and $Z : \mathbb{Z}^n \rightarrow \mathbb{C}$ is defined by $Z(e_i) = z_i$, the component of $\phi(H)_n((x_1 + \cdots + x_n)^{\otimes n})$ of weight $e_1 + \cdots + e_n$ is equal to

$$\sum_{\sigma \in \text{Sym}_n} H_n(z_{\sigma(1)}, \dots, z_{\sigma(n)}) x_{\sigma(1)} \cdots x_{\sigma(n)}$$

The statements (i) and (ii) now follow from the foregoing and Lemma 11.5 since $\phi(G \circ F) = \phi(G) \circ \phi(F)$ and $\phi(F^{-1}) = \phi(F)^{-1}$. \square

11.10. We shall in fact need to consider transforms whose associated functions $F_n : \mathbb{C}^n \rightarrow \mathbb{C}$ with $n \geq 2$ are only defined on $(\mathbb{C}^*)^n$ and satisfy

$$F_n(z_1, \dots, z_n) = 0 \quad \text{whenever} \quad z_1 + \cdots + z_n = 0 \quad (57)$$

We will in such cases tacitly extend the functions F_n to \mathbb{C}^n in an arbitrary way. This does not affect the values of the composition $G \circ F$ of two such transforms on $\bigsqcup_{n \geq 1} (\mathbb{C}^*)^n$ since, by (54) any evaluation of G_k at an argument $w_{i_k} = \sum_{i=i_{k-1}+1}^{i_k} z_i$ is multiplied by $F_{i_k-i_{k-1}}(z_{i_{k-1}+1}, \dots, z_{i_k})$ which vanishes if w_{i_k} does. Moreover, the composition $G \circ F$ satisfies (57) if F and G do.

Similarly, if F is an invertible transform satisfying these properties, the inverse transform F^{-1} satisfies (57), and the values of $(F^{-1})_n$ on $(\mathbb{C}^*)^n$ do not depend on the extension of the functions F_m to \mathbb{C}^m .

Let now (A, Λ, Z) be a graded algebra as in 11.7, and consider the subspace $A_Z^\times \subset A$ defined by

$$A_Z^\times = \bigoplus_{\substack{\lambda \in \Lambda \\ Z(\lambda) \neq 0}} A_\lambda.$$

If F is a transform defined on $\bigsqcup_{n \geq 1} (\mathbb{C}^*)^n$, the restriction of $\phi(F)_n$ to $(A_Z^\times)^{\otimes n}$ is clearly independent of the extension of F_n to \mathbb{C}^n . Moreover, if F satisfies (57), the image of $\phi(F)_n$ lies in A_Z^\times for $n \geq 2$, so that $\phi(F)$ restricts to a NC power series $A_Z^\times \rightarrow A_Z^\times$.

11.11. Let U and V be finite-dimensional and $\phi: U \rightarrow V$ a commutative power series. Assume that the sum

$$\underline{\phi}(u) = \sum_{n \geq 1} \phi_n(u^{\otimes n}) \quad (58)$$

is convergent for all u in an open neighbourhood of the origin $0 \in U^o \subset U$. Then $\underline{\phi}$ defines a holomorphic map $U^o \rightarrow V$ and (58) is its Taylor expansion at the origin.

If $\phi: U \rightarrow V$ and $\psi: V \rightarrow W$ are two commutative power series which convergent in neighbourhoods of the origins in U and V respectively then the power series $\psi \circ \phi$ is convergent in a neighbourhood of the origin in U and $\underline{(\psi \circ \phi)} = \underline{\psi} \circ \underline{\phi}$.

Similarly, if V is finite-dimensional and $\phi: V \rightarrow V$ is an invertible power series which is convergent in a neighbourhood of the origin, then by the inverse function theorem, the inverse map $\underline{\phi}^{-1}$ is holomorphic near the origin. The following is standard.

Lemma. *The power series ϕ^{-1} is convergent in a neighborhood of the origin in V and $\underline{\phi}^{-1} = \underline{\phi}^{-1}$.*

PROOF. If $f: V \rightarrow V$ is a germ of a holomorphic function at $0 \in V$ such that $f(0) = 0$, we denote its Taylor series, viewed as a commutative power series, by Tf . Thus, $\underline{Tf} = f$ and $T(\underline{\phi}) = \phi$ whenever $\underline{\phi}$ is defined. Since $\underline{\phi} \circ \underline{\phi}^{-1} = \text{id}_V = \underline{\phi}^{-1} \circ \underline{\phi}$ we find, upon applying T that

$$\phi \circ T(\underline{\phi}^{-1}) = \text{id}_V = T(\underline{\phi}^{-1}) \circ \phi$$

Thus $\phi^{-1} = T(\underline{\phi}^{-1})$ as claimed. \square

11.12. **Proof of Theorem 4.8.** We shall proceed by extending the Taylor series of the Stokes map \mathcal{S} to a commutative power series $U\mathfrak{g} \rightarrow U\mathfrak{g}$, then lift it to a NC Lie transform $\phi(L): U\mathfrak{g} \rightarrow U\mathfrak{g}$ and finally invert it by using Theorem 11.3. The Taylor series of \mathcal{S}^{-1} will then be obtained as the restriction to $\mathfrak{g}_{\text{od}} \subset U\mathfrak{g}$ of the commutative power series $\sigma^* \phi(L)^{-1}$.

Specifically, let $U\mathfrak{g}$ be the universal enveloping algebra of \mathfrak{g} , graded by the lattice $\Lambda \subset \mathfrak{h}^*$ spanned by the set of roots $\Phi(G; H)$. Let $L_n: (\mathbb{C}^*)^n \rightarrow \mathbb{C}$ be the functions defined in 4.7, $L = \{L_n\}_{n \geq 1}$ the corresponding transform and $\phi(L): U\mathfrak{g} \rightarrow U\mathfrak{g}$

the NC power series determined by $(U\mathfrak{g}, \Lambda, Z)$ and L . Thus, for any $n \geq 1$ and $x^1, \dots, x^n \in U\mathfrak{g}$,

$$\phi(L)_n(x^1 \otimes \cdots \otimes x^n) = \sum_{\gamma_1, \dots, \gamma_n} L_n(Z(\gamma_1), \dots, Z(\gamma_n)) x_{\gamma_1}^1 \cdots x_{\gamma_n}^n \quad (59)$$

where x_γ is the weight component of x corresponding to $\gamma \in \Lambda$.

The power series $\phi(L)$ preserves the subspace $\mathfrak{g} \subset U\mathfrak{g}$ in the sense of 11.5 since L is a Lie transform by Theorem 4.7. By 11.10, each $\phi(L)_n$, $n \geq 2$, maps $(U\mathfrak{g})^{\otimes n}$ to

$$U\mathfrak{g}^\times = \bigoplus_{\substack{\lambda \in \Lambda \\ Z(\lambda) \neq 0}} (U\mathfrak{g})_\lambda$$

since L satisfies (57) by (22). Thus, $\phi(L)_n$ maps $S^n \mathfrak{g}_{\text{od}}$ to $\mathfrak{g}_{\text{od}} = U\mathfrak{g}^\times \cap \mathfrak{g}$ so that the restriction of $\sigma^* \phi(L)$ to $\mathfrak{g}_{\text{od}} \subset U\mathfrak{g}$ is a commutative power series $\mathfrak{g}_{\text{od}} \rightarrow \mathfrak{g}_{\text{od}}$ which is equal to the Taylor series of the Stokes map \mathcal{S} by Theorem 4.7.

Let now $J = L^{-1}$ be the inverse transform. By Proposition 11.8, $J_1 \equiv (2\pi i)^{-1}$ and, for $n \geq 2$,

$$J_n(z_1, \dots, z_n) = (2\pi i)^{-n} \sum_T (-1)^{|V(T)|} J_T(z_1, \dots, z_n)$$

where the sum is over rooted plane trees with n leaves all of whose vertices have valency ≥ 3 and J_T is defined by (56). J is a Lie transform by Lemma 11.9 which satisfies (57) since L does. Thus $\sigma^* \phi(J) = \sigma^* \phi(L)^{-1}$ restricts to a commutative power series $\mathfrak{g}_{\text{od}} \rightarrow \mathfrak{g}_{\text{od}}$. The latter is given by (25) and is equal to the Taylor series of \mathcal{S}^{-1} at $\epsilon = 0$ by Lemma 11.11. \square

Remark. The transform $\phi(L)$ is in a sense the most economical lift of the Taylor series of \mathcal{S} to a NC power series. In particular, it differs from the more canonical lift obtained by composing the terms $S^n \mathfrak{g}_{\text{od}} \rightarrow \mathfrak{g}_{\text{od}}$ of the Taylor series of \mathcal{S} with the canonical projection $\mathfrak{g}_{\text{od}}^{\otimes n} \rightarrow S^n \mathfrak{g}_{\text{od}}$.

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