



An introduction to motivic Hall algebras

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Abstract

We give an introduction to Joyce's construction of the motivic Hall algebra of coherent sheaves on a variety M . When M is a Calabi–Yau threefold we define a semi-classical integration map from a Poisson subalgebra of this Hall algebra to the ring of functions on a symplectic torus. This material will be used in Bridgeland (2011) [3] to prove some basic properties of Donaldson–Thomas curve-counting invariants on Calabi–Yau threefolds.

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1. Introduction

This paper is a gentle introduction to part of Joyce's theory of motivic Hall algebras [9–12]. It started life as the first half of the author's paper [3] in which this theory is used to prove some properties of Donaldson–Thomas curve-counting invariants on Calabi–Yau threefolds. Eventually it became clear that there were enough points at which our presentation differs from Joyce's to justify a separate paper. Nonetheless, most of the basic ideas can be found in Joyce's work.

The application of Hall algebras to the study of invariants of moduli spaces originated with Reineke's computation of the Betti numbers of the spaces of stable quiver representations [20]. His technique was to translate categorical statements into identities in a suitable Hall algebra, and to then apply a ring homomorphism into a completed skew-polynomial ring, thus obtaining identities involving generating functions for the invariants of interest.

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The relevant category in Reineke’s paper is the category of representations of a finite quiver without relations. Such categories can be defined over any field, and Reineke worked with a Hall algebra based on counting points over \mathbb{F}_q . In [11] Joyce used Grothendieck rings of Artin stacks to construct a motivic version of the Hall algebra defined in arbitrary characteristic. This can be applied, for example, to categories of coherent sheaves on complex varieties.

The interesting part of the theory is the construction of a homomorphism from the Hall algebra to a skew polynomial ring, often viewed as a ring of functions on a quantum torus. Such maps go under the general name of integration maps, since they involve integrating an element of the Hall algebra over the moduli stack. In Reineke’s case the existence of such a map relied on the fact that the relevant categories of representations were of homological dimension 1. Remarkably it seems that integration maps also exist when the underlying abelian category is Calabi–Yau of dimension 3.

In the CY_3 case Joyce’s integration map is a homomorphism of Lie algebras defined on a Lie subalgebra of the Hall algebra. Kontsevich and Soibelman [14] suggested that incorporating motivic vanishing cycles should enable one to construct an algebra morphism from the full Hall algebra. Unfortunately the details of the constructions in their paper are currently rather sketchy and rely on some unproved conjectures. Joyce and Song [12] went on to use some of the ideas from [14] to prove an important property of Behrend functions (stated here as Theorem 5.3) and so incorporate such functions into Joyce’s Lie algebra integration map.

The main result of this paper (Theorem 5.2) is the existence of an integration map in the CY_3 case that is a homomorphism of Poisson algebras. It can be viewed as the semi-classical limit of the ring homomorphism envisaged by Kontsevich and Soibelman. It relies on the same property of the Behrend function proved by Joyce and Song. Combined with a difficult no-poles result of Joyce it can be used to prove non-trivial results on Donaldson–Thomas invariants [3].

2. Grothendieck rings of varieties and schemes

Here we review some basic definitions concerning Grothendieck rings of varieties. Some good references are [2,17]. For us a complex variety is a reduced, separated scheme of finite type over \mathbb{C} . We denote by

$$\text{Var}/\mathbb{C} \subset \text{Sch}/\mathbb{C} \subset \text{Sp}/\mathbb{C} \quad (1)$$

the categories of varieties over \mathbb{C} , of schemes of finite type over \mathbb{C} , and of algebraic spaces of finite type over \mathbb{C} respectively.

2.1. Grothendieck ring of varieties

Recall first the definition of the Grothendieck ring of varieties.

Definition 2.1. Let $K(\text{Var}/\mathbb{C})$ denote the free abelian group on isomorphism classes of complex varieties, modulo relations

$$[X] = [Z] + [U] \quad (2)$$

for $Z \subset X$ a closed subvariety with complementary open subvariety U .

The relations (2) are called the scissor relations, since they involve cutting a variety up into pieces. We can equip $K(\text{Var}/\mathbb{C})$ with the structure of a commutative ring by setting

$$[X] \cdot [Y] = [X \times Y].$$

The class of a point $1 = [\text{Spec}(\mathbb{C})]$ is then a unit. We write

$$\mathbb{L} = [\mathbb{A}^1] \in K(\text{Var}/\mathbb{C})$$

for the class of the affine line. By a stratification of a variety X we mean a collection of disjoint locally-closed subsets $X_i \subset X$ which together cover X .

Lemma 2.2. *If a variety X is stratified by subvarieties X_i then only finitely many of the X_i are non-empty and*

$$[X] = \sum_i [X_i] \in K(\text{Var}/\mathbb{C}).$$

Proof. The result is clear for varieties of dimension 0, so let us use induction on the dimension d of X , and assume the result known for varieties of dimension $< d$.

First consider the case when X is irreducible. Then one of the $X_i = U$ contains the generic point and is therefore open. The complement $Z = X \setminus U$ is of smaller dimension and is stratified by the other subvarieties X_i . Since

$$[X] = [Z] + [U]$$

the result follows by induction.

In the case when X is reducible we can take an irreducible subvariety and remove the intersections with the other irreducible components. This gives an irreducible open subset $U \subset X$ with complement a closed subvariety Z having fewer irreducible components than X . By induction on this number one can therefore conclude that

$$[Z] = \sum_i [Z \cap X_i], \quad [U] = \sum_i [U \cap X_i],$$

with finitely many non-empty terms appearing in each sum. Since

$$[X_i] = [Z \cap X_i] + [U \cap X_i],$$

the result then follows from the scissor relations. \square

There is a ring homomorphism $\chi : K(\text{Var}/\mathbb{C}) \rightarrow \mathbb{Z}$ defined by sending the class of a variety X to its topological Euler characteristic

$$\chi(X) = \sum_{i=0}^{2d} (-1)^i \dim H^i(X_{\text{an}}, \mathbb{C}),$$

where X_{an} denotes X equipped with the analytic topology, and H^i denotes singular cohomology.

Lemma 2.3. *Suppose $x \in K(\text{Var}/\mathbb{C})$ satisfies*

$$\mathbb{L}^m \cdot (\mathbb{L}^n - 1) \cdot x = 0$$

for some $m, n \geq 1$. Then $\chi(x) = 0$.

Proof. There is a ring homomorphism $\chi_t : K(\text{Var}/\mathbb{C}) \rightarrow \mathbb{Z}[t]$ that sends the class of a smooth complete variety X to the Poincaré polynomial

$$\chi_t(X) = \sum_{i=0}^{2d} t^i \cdot \dim H^i(X_{\text{an}}, \mathbb{C}).$$

It specialises at $t = -1$ to the Euler characteristic. Now

$$\chi_t(\mathbb{L}) = \chi_t(\mathbb{P}^1) - \chi_t(1) = t^2.$$

Since $\mathbb{Z}[t]$ is an integral domain one therefore has $\chi_t(x) = 0$. Setting $t = -1$ gives the result. \square

2.2. Zariski fibrations

There is a useful identity in $K(\text{Var}/\mathbb{C})$ relating to fibrations.

Definition 2.4. A morphism of schemes $f : X \rightarrow Y$ will be called a Zariski fibration if there is an open cover $Y = \bigcup_{i \in I} U_i$ and diagrams

$$\begin{array}{ccc} f^{-1}(U_i) & \xrightarrow{g_i} & U_i \times F_i \\ & \searrow f & \swarrow \pi_1 \\ & U_i & \end{array}$$

with each g_i an isomorphism.

Of course if Y is connected then all the fibres F_i are isomorphic, but this will often not be the case. We say that two Zariski fibrations

$$f_1 : X_1 \rightarrow Y \quad \text{and} \quad f_2 : X_2 \rightarrow Y$$

have the same fibres, if for any point $y \in Y(\mathbb{C})$ the fibres of f_1 and f_2 over y are isomorphic.

Lemma 2.5. *Suppose $f_1 : X_1 \rightarrow Y$ and $f_2 : X_2 \rightarrow Y$ are Zariski fibrations of varieties with the same fibres. Then*

$$[X_1] = [X_2] \in K(\text{Var}/\mathbb{C}).$$

Proof. We can stratify Y by a finite collection of connected, locally-closed subvarieties $Y_i \subset Y$ such that f_1 and f_2 are trivial fibrations over each Y_i . Then

$$[X_1] = \sum_i [f_1^{-1}(Y_i)] = \sum_i [F_i] \cdot [Y_i] = \sum_i [f_2^{-1}(Y_i)] = [X_2],$$

where F_i is the common fibre of f_1 and f_2 over Y_i . \square

The following application of Lemma 2.5 will be important later.

Lemma 2.6. *There is an identity*

$$[\mathrm{GL}_d] = \mathbb{L}^{\frac{1}{2}d(d-1)} \cdot \prod_{k=1}^d (\mathbb{L}^k - 1) \in K(\mathrm{Var}/\mathbb{C}).$$

Proof. Let $B \subset \mathrm{GL}_d$ be the stabiliser of a non-zero vector $x \in \mathbb{C}^d$. The assignment $g \mapsto g(x)$ defines a morphism

$$\pi : \mathrm{GL}_d \rightarrow \mathbb{C}^d \setminus \{0\}$$

which is easily seen to be a Zariski fibration with fibre B . But there is also an isomorphism

$$B \cong \mathrm{GL}_{d-1} \times \mathbb{C}^{d-1}.$$

Thus by Lemma 2.5

$$[\mathrm{GL}_d] = (\mathbb{L}^d - 1) \cdot \mathbb{L}^{d-1} \cdot [\mathrm{GL}_{d-1}],$$

and the result follows by induction. \square

2.3. Geometric bijections

We will base our treatment of Grothendieck groups on the following class of maps.

Definition 2.7. A morphism $f : X \rightarrow Y$ in the category Sch/\mathbb{C} is a geometric bijection if it induces a bijection

$$f(\mathbb{C}) : X(\mathbb{C}) \rightarrow Y(\mathbb{C})$$

between the sets of \mathbb{C} -valued points.

Using Lemma 2.8 below, it is not difficult to prove that the condition of Definition 2.7 is equivalent to f being a bijection, or a universal bijection, but for various reasons we prefer to introduce new terminology.

By a stratification of a scheme X we mean a collection of disjoint locally-closed subschemes $X_i \subset X$ which together cover X . If the scheme X is of finite type over \mathbb{C} then the argument of Lemma 2.2 shows that only finitely many of the X_i can be non-empty.

Lemma 2.8. *A morphism $f : X \rightarrow Y$ in the category Sch/\mathbb{C} is a geometric bijection precisely if there are stratifications*

$$X_i \subset X, \quad Y_i \subset Y,$$

such that f induces isomorphisms $f_i : X_i \rightarrow Y_i$.

Proof. A more precise statement of the condition is that there should be isomorphisms $f_i : X_i \rightarrow Y_i$ and commuting diagrams

$$\begin{array}{ccc} X_i & \xrightarrow{f_i} & Y_i \\ j_i \downarrow & & \downarrow k_i \\ X & \xrightarrow{f} & Y \end{array}$$

where the morphisms j_i and k_i are the embeddings of the given locally-closed subschemes. This condition is clearly sufficient since every \mathbb{C} -valued point of X or Y factors through a unique one of the given subschemes.

For the converse we may as well assume that X and Y are reduced, since a stratification of X_{red} also gives a stratification of X , and similarly for Y . We claim that there is an open subscheme $Y_1 \subset Y$ such that f induces an isomorphism

$$f : X_1 \rightarrow Y_1,$$

where $X_1 = f^{-1}(Y_1)$. This will be enough since we can then replace X and Y by the complements of X_1 and Y_1 and repeat.

To prove the claim we can pass to an open subset of Y and hence assume that Y is an irreducible variety. By generic flatness, we can also assume that f is flat, and hence open. Now we can replace X by an irreducible open subvariety, and so f becomes a map of irreducible varieties. The claim then holds by [19, Proposition 3.17]. \square

Using Lemma 2.8 we can give an alternative definition of the Grothendieck ring in terms of bijections. This is sometimes useful, particularly when considering Grothendieck rings of schemes and stacks of possibly infinite type.

Lemma 2.9. *The group $K(\text{Var}/\mathbb{C})$ is the free abelian group on isomorphism classes of the category Var/\mathbb{C} , modulo relations*

- (a) $[X_1 \amalg X_2] = [X_1] + [X_2]$ for every pair of varieties X_1 and X_2 ,
- (b) $[X] = [Y]$ for every geometric bijection $f : X \rightarrow Y$.

Proof. Given a geometric bijection of varieties $f : X \rightarrow Y$, we can take stratifications of X and Y as in Lemma 2.8. We can always assume that the subschemes X_i and Y_i are reduced and hence subvarieties. Then, by Lemma 2.2,

$$[X] = \sum_i [X_i] = \sum_i [Y_i] = [Y] \in K(\text{Var}/\mathbb{C}).$$

Thus relation (b) is a consequence of the scissor relations, and clearly relation (a) is a special case of them. Conversely, given a decomposition as in Definition 2.1, the obvious morphism $Z \sqcup U \rightarrow X$ is a geometric bijection, so the scissor relations of Definition 2.1 are a consequence of the relations (a) and (b) in the statement of the lemma. \square

2.4. *Grothendieck rings of schemes and algebraic spaces*

Before considering stacks in the next section, it is worth briefly considering the case of schemes and algebraic spaces, always of finite type over \mathbb{C} .

Definition 2.10. Let $K(\text{Sch}/\mathbb{C})$ be the free abelian group on isomorphism classes of the category Sch/\mathbb{C} , modulo relations

- (a) $[X_1 \sqcup X_2] = [X_1] + [X_2]$ for every pair of schemes X_1 and X_2 ,
- (b) $[X] = [Y]$ for every geometric bijection $f : X \rightarrow Y$.

The product in Sch/\mathbb{C} gives the group $K(\text{Sch}/\mathbb{C})$ the structure of a commutative ring. One could alternatively define $K(\text{Sch}/\mathbb{C})$ via scissor relations as in Definition 2.1; the argument of Lemma 2.9 shows that this would give the same ring.

In the case of algebraic spaces we define geometric bijections in the category Sp/\mathbb{C} exactly as in Definition 2.7. We can also define the notion of a stratification of an algebraic space in the obvious way.

Lemma 2.11. *A morphism $f : X \rightarrow Y$ in the category Sp/\mathbb{C} is a geometric bijection precisely if there are stratifications*

$$X_i \subset X, \quad Y_i \subset Y,$$

such that f induces isomorphisms $f_i : X_i \rightarrow Y_i$.

Proof. This follows the same lines as that of Lemma 2.8. The extra argument needed is the following. Suppose given a morphism $f : X \rightarrow Y$ in Sp/\mathbb{C} with X and Y reduced. We must show that there is an open subspace $Y_1 \subset Y$ such that f induces an isomorphism

$$f : X_1 \rightarrow Y_1,$$

where $X_1 = f^{-1}(Y_1)$. By [13, Proposition II.6.6] we can pass to an open subset and so assume that Y is a scheme, and even an irreducible variety. By generic flatness we can also assume that f is flat. Then, using the same result from [13] again, there is an open subset $X_0 \subset X$ representable by an irreducible variety. Since the induced map $f : X_0 \rightarrow Y$ is flat, its image is an open subvariety $Y_0 \subset Y$. We can then apply [19, Proposition 3.17] as in the proof of Lemma 2.8. \square

The Grothendieck group $K(\text{Sp}/\mathbb{C})$ is defined as in Definition 2.10, replacing the category Sch/\mathbb{C} by Sp/\mathbb{C} . The following result shows that from the point of view of Grothendieck rings, providing we stick to objects of finite type, the distinction between varieties, schemes and algebraic spaces disappears.

Lemma 2.12. *The embeddings of categories (1) induce isomorphisms of rings*

$$K(\text{Var}/\mathbb{C}) \cong K(\text{Sch}/\mathbb{C}) \cong K(\text{Sp}/\mathbb{C}).$$

Proof. The basic point is that if $Y \in \text{Sp}/\mathbb{C}$ there is a geometric bijection

$$f : X \rightarrow Y$$

with X a variety. Indeed, by [13, Proposition II.6.6] there is an open subspace $U \subset Y$ that is representable by an affine scheme. Taking the complement and repeating, we can stratify Y by affine schemes $Y_i \subset Y$. The inclusion map from the disjoint union of these strata then defines a geometric bijection

$$X = \coprod Y_i \rightarrow Y$$

with X an affine scheme of finite type over \mathbb{C} . We can assume that X is reduced and hence a variety since the inclusion of its reduced subscheme is another geometric bijection.

Now consider the homomorphism

$$I : K(\text{Var}/\mathbb{C}) \rightarrow K(\text{Sp}/\mathbb{C})$$

induced by the inclusion of varieties in algebraic spaces. By the above it is surjective, so it will be enough to construct a left inverse

$$P : K(\text{Sp}/\mathbb{C}) \rightarrow K(\text{Var}/\mathbb{C}).$$

Given an object $Y \in \text{Sp}/\mathbb{C}$, take a geometric bijection $f : X \rightarrow Y$ with X a variety, and set

$$P([Y]) = [X].$$

This is well-defined, because if W is another variety with a geometric bijection $g : W \rightarrow Y$, then we can form the fibre square

$$\begin{array}{ccc} Z & \xrightarrow{p} & W \\ q \downarrow & & \downarrow g \\ X & \xrightarrow{f} & Y \end{array}$$

and taking a variety T with a geometric bijection $g : T \rightarrow Z$, the composite morphisms $p \circ g$ and $q \circ g$ are geometric bijections, so

$$[X] = [W] \in K(\text{Var}/\mathbb{C}).$$

It is easy to check that P preserves the relations and hence defines the required inverse. \square

3. Grothendieck rings of stacks

In this section we consider Grothendieck rings of algebraic stacks. The author learnt most of the material in this section from the papers of Joyce [9] and Toën [21]. We refer to [7] for a readable introduction to stacks, and to [16] for a more detailed treatment. All stacks will be Artin stacks and will be assumed to be locally of finite type over \mathbb{C} . We denote by St/\mathbb{C} the 2-category of algebraic stacks of finite type over \mathbb{C} .

Given a scheme S and a stack X we denote by $X(S)$ the groupoid of S -valued points of X . We will be fairly sloppy with 2-terminology: by a commutative (resp. Cartesian) diagram of stacks we mean one that is 2-commutative (resp. 2-Cartesian). By an isomorphism of stacks we mean what a category-theorist would call an equivalence.

3.1. Geometric bijections and Zariski fibrations

In the case of stacks the appropriate analogue of Definition 2.7 is as follows.

Definition 3.1. A morphism $f : X \rightarrow Y$ in the category St/\mathbb{C} will be called a geometric bijection if it is representable and the induced functor on groupoids of \mathbb{C} -valued points

$$f(\mathbb{C}) : X(\mathbb{C}) \rightarrow Y(\mathbb{C})$$

is an equivalence of categories.

In fact, it is easy enough to show that the assumption that f be representable in Definition 3.1 follows from the other condition. This comes down to showing that if $f : X \rightarrow Y$ is a group scheme in Sch/\mathbb{C} such that for each point $y \in Y(\mathbb{C})$ the fibre X_y is the trivial group, then f is an isomorphism.

By a stratification of a stack X we mean a collection of locally-closed substacks $X_i \subset X$ that are disjoint and together cover X . If X is of finite type over \mathbb{C} it follows by pulling back to an atlas that only finitely many of the X_i can be non-empty.

Lemma 3.2. A morphism $f : X \rightarrow Y$ in St/\mathbb{C} is a geometric bijection precisely if there are stratifications

$$X_i \subset X, \quad Y_i \subset Y,$$

such that f induces isomorphisms $f_i : X_i \rightarrow Y_i$.

Proof. As in the proof of Lemma 2.8 it is easy to see that the condition is sufficient. For the converse, assume that f is a geometric bijection. Replacing X and Y by their reduced substacks we can assume that X and Y are reduced. It will be enough to show that there is a non-empty open substack $Y_1 \subset Y$ such that f induces an isomorphism

$$f : X_1 \rightarrow Y_1,$$

where $X_1 = f^{-1}(Y_1)$. We can then take complements and repeat.

Pulling f back to an atlas $\pi : T \rightarrow Y$ we obtain a diagram

$$\begin{array}{ccc} S & \xrightarrow{g} & T \\ \downarrow & & \downarrow \pi \\ X & \xrightarrow{f} & Y \end{array}$$

with S an algebraic space and g a geometric bijection.

Let $T_1 \subset T$ be the largest open subset over which g is an isomorphism. By Lemma 2.11 this subset is non-empty. That it descends to Y follows from the following statement. Suppose given a Cartesian diagram in Sp/\mathbb{C}

$$\begin{array}{ccc} S' & \xrightarrow{g'} & T' \\ q \downarrow & & \downarrow p \\ S & \xrightarrow{g} & T \end{array} \tag{3}$$

with p faithfully flat. Then g' is an isomorphism over an open subset $U' \subset T'$ precisely if g is an isomorphism over the open subset $U = p(U') \subset T$.

Replacing T by U and T' by U' we can reduce further to the statement that given a diagram (3) with p faithfully flat and g' an isomorphism then g is also an isomorphism. This is the well-known statement that isomorphisms are stable in the faithfully flat topology [13, Proposition 3.5]. \square

Definition 3.3. A morphism of stacks $f : X \rightarrow Y$ is a Zariski fibration if its pullback to any scheme is a Zariski fibration of schemes.

In particular a Zariski fibration of stacks is representable. Note however that there need not be a cover of the stack Y by open substacks over which f is trivial. For this reason, in the Grothendieck group of stacks, the fibration identity of Lemma 2.5 is not a consequence of the other relations, and we will impose it by hand.

3.2. Grothendieck ring of stacks

To make the comparison result in the next section true, it will be necessary to restrict slightly the class of stacks we allow in our Grothendieck ring.

Definition 3.4. A stack X locally of finite type over \mathbb{C} has affine stabilisers if for every \mathbb{C} -valued point $x \in X(\mathbb{C})$ the group $\text{Isom}_{\mathbb{C}}(x, x)$ is affine.

The importance of this notion lies in the following corollary of a result of Kresch.

Proposition 3.5. A stack $X \in \text{St}/\mathbb{C}$ has affine stabilisers precisely if there is a variety Y with an action of $G = \text{GL}_d$ and a geometric bijection

$$f : Y/G \rightarrow X.$$

Proof. The condition is clearly sufficient. For the converse, suppose first that the automorphism groups of all geometric points of X are affine. Kresch then shows [15, Proposition 3.5.2, Proposition 3.5.9] that the associated reduced stack X_{red} has a stratification by locally-closed substacks of the form

$$X_i = Y_i/G_i,$$

where by [15, Lemma 3.5.1] we can take each group G_i to be of the form $G_i = \text{GL}_{d_i}$. The obvious map $f : \sqcup X_i \rightarrow X$ is then a geometric bijection, and the result follows from the isomorphism

$$(X_1/G_1) \sqcup (X_2/G_2) \cong [(X_1 \times G_2) \sqcup (X_2 \times G_1)]/(G_1 \times G_2),$$

which shows that the disjoint union of quotient stacks is another quotient stack.

To finish the proof we must check that if the automorphism groups of all \mathbb{C} -valued points of X are affine, then the same is true for all geometric points. To prove this,¹ suppose $f : G \rightarrow S$ is a group scheme, with G and S of finite type over \mathbb{C} , and suppose that there is a geometric point $\text{Spec}(K) \rightarrow S$ such that the corresponding geometric fibre of f is non-affine. We must prove that there is a \mathbb{C} -valued point of S such that the corresponding fibre of f is non-affine.

Since being affine is invariant under field extension [EGA IV.2.7.1] we can assume that K is the algebraic closure of the residue field $k(s)$ of a point $s \in S$. Chevalley’s theorem [5] implies that an algebraic group G over a field K of characteristic zero is non-affine precisely if there is an epimorphism $G_0 \twoheadrightarrow A$ where $G_0 \subset G$ is the connected component of the identity, and A is a positive-dimensional abelian variety. For a modern proof of this see [6].

Applying this to our situation, the epimorphism $G_0 \twoheadrightarrow A$ is defined over a finite extension of $k(s)$, and hence over a finite type scheme with a morphism $T \rightarrow S$ dominating the closure of the point s . Restricting to a \mathbb{C} -valued point and applying Chevalley’s theorem again completes the proof. \square

We can now give the following definition.

Definition 3.6. Let $K(\text{St}/\mathbb{C})$ be the free abelian group spanned by isomorphism classes of stacks of finite type over \mathbb{C} with affine stabilisers, modulo relations

- (a) $[X_1 \sqcup X_2] = [X_1] + [X_2]$ for every pair of stacks X_1 and X_2 ,
- (b) $[X] = [Y]$ for every geometric bijection $f : X \rightarrow Y$,
- (c) $[X_1] = [X_2]$ for every pair of Zariski fibrations $f_i : X_i \rightarrow Y$ with the same fibres.

Fibre product of stacks over \mathbb{C} gives $K(\text{St}/\mathbb{C})$ the structure of a commutative ring. There is an obvious homomorphism of commutative rings

$$K(\text{Var}/\mathbb{C}) \rightarrow K(\text{St}/\mathbb{C}) \tag{4}$$

obtained by considering a variety as a representable stack.

¹ The author is grateful to Andrew Kresch for explaining this argument.

3.3. Comparison lemma

In this section, following Toën [21], we show that if one localises $K(\text{Var}/\mathbb{C})$ at the classes of all special algebraic groups, the map (4) becomes an isomorphism.

Definition 3.7. An algebraic group is special if any map of schemes

$$f : X \rightarrow Y$$

which is a principal G -bundle in the étale topology is a Zariski fibration.

Examples of special groups include the general linear groups GL_d . All special groups are affine and connected [4].

Lemma 3.8. *Localising the ring $K(\text{Var}/\mathbb{C})$ with respect to any of the following three sets of elements gives the same result:*

- (a) *the classes $[G]$ for G a special algebraic group,*
- (b) *the classes $[\text{GL}_d]$ for $d \geq 1$,*
- (c) *the elements \mathbb{L} and $\mathbb{L}^i - 1$ for $i \geq 1$.*

Proof. If $G \subset \text{GL}_d$ is a closed subgroup, then the quotient map

$$\pi : \text{GL}_d \rightarrow \text{GL}_d/G$$

is a principal G -bundle. It is locally trivial in the smooth topology (equivalently, in the étale topology [23, Example 2.51]) because it is smooth, and becomes trivial when pulled back to itself. Thus if G is special we can conclude that

$$[\text{GL}_d] = [G] \cdot [\text{GL}_d/G].$$

Since GL_d is itself special this proves the equivalence of (a) and (b). The equivalence of (b) and (c) follows from Lemma 2.5. \square

If an algebraic group G acts on a variety X then the map to the quotient stack

$$\pi : X \rightarrow X/G$$

is a principal G -bundle when pulled back to any scheme. If G is special it follows that π is a Zariski fibration with fibre G and hence

$$[X] = [G] \cdot [X/G] \in K(\text{St}/\mathbb{C}). \tag{5}$$

In particular taking X to be a point we see that $[G]$ is invertible in $K(\text{St}/\mathbb{C})$.

The following result is due to Toën [21, Theorem 3.10] (who also considered higher stacks). A slightly weaker version was proved independently by Joyce [9, Theorem 4.10].

Lemma 3.9. *The homomorphism (4) induces an isomorphism of commutative rings*

$$Q : K(\text{Var}/\mathbb{C})[[\text{GL}(d)]^{-1} : d \geq 1] \longrightarrow K(\text{St}/\mathbb{C}).$$

Proof. The expression (5) shows that the map Q is well-defined and satisfies

$$Q([X]/[G]) = [X/G] \in K(\text{St}/\mathbb{C}).$$

We shall construct an inverse to Q . Suppose Z is a stack with affine stabilisers. By Proposition 3.5 there is a geometric bijection $g : X/G \rightarrow Z$ with G a special algebraic group acting on a variety X . Set

$$R(Z) = [X]/[G] \in K(\text{Var}/\mathbb{C})[[\text{GL}(d)]^{-1} : d \geq 1].$$

To check that this is well-defined, suppose given another bijection $h : Y/H \rightarrow Z$. Consider the diagram of Cartesian squares

$$\begin{array}{ccccc} R & \xrightarrow{p} & Q & \xrightarrow{j} & Y \\ q \downarrow & & \downarrow & & \downarrow \\ P & \longrightarrow & W & \longrightarrow & Y/H \\ k \downarrow & & \downarrow & & \downarrow h \\ X & \xrightarrow{\pi} & X/G & \xrightarrow{g} & Z \end{array}$$

Then Q is an algebraic space and j is a geometric bijection, so $[Q] = [Y] \in K(\text{Sp}/\mathbb{C})$. The map π , and hence also p , is a Zariski fibration with fibre G ; pulling it back to a bijection $Q' \rightarrow Q$ with Q' a scheme, it follows that $[R] = [G] \cdot [Q] \in K(\text{Sp}/\mathbb{C})$. Hence we obtain $[R] = [G] \cdot [Y]$ and, by symmetry, $[R] = [H] \cdot [X]$. Thus, using Lemma 2.12

$$[G] \cdot [Y] = [H] \cdot [X] \in K(\text{Var}/\mathbb{C}).$$

To show that R descends to the level of the Grothendieck group we must check that the relations in Definition 3.6 are mapped to zero. This is very easy for (a) and (b). For (c) suppose $f_i : X_i \rightarrow Y$ are Zariski fibrations with the same fibres. Take a geometric bijection $W/G \rightarrow Y$ and form the diagrams

$$\begin{array}{ccccc} S_i & \xrightarrow{k_i} & T_i & \xrightarrow{h_i} & X_i \\ p_i \downarrow & & \downarrow & & \downarrow f_i \\ W & \xrightarrow{\pi} & W/G & \xrightarrow{g} & Y \end{array}$$

Then there are induced actions of G on the varieties S_i such that $T_i \cong S_i/G$. Since h_i is a geometric bijection, $R(X_i) = [S_i]/[G]$. On the other hand, by pullback, the morphisms $p_i : S_i \rightarrow W$ are Zariski fibrations of schemes with the same fibres, and hence by the argument of Lemma 2.5, $[S_1] = [S_2] \in K(\text{Sch}/\mathbb{C})$. \square

3.4. Relative Grothendieck groups

Let S be a fixed algebraic stack, locally of finite type over \mathbb{C} . We shall always assume that S has affine stabilisers. There is a 2-category of algebraic stacks over S . Let St/S denote the full subcategory consisting of objects

$$f : X \rightarrow S \tag{6}$$

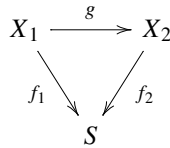
for which X is of finite type over \mathbb{C} . Such an object will be said to have affine stabilisers if the stack X has. Repeating Definition 3.6 in this relative context gives the following.

Definition 3.10. Let $K(\text{St}/S)$ be the free abelian group spanned by isomorphism classes of objects (6) of St/S , with affine stabilisers, modulo relations

(a) for every pair of objects X_1 and X_2 a relation

$$[X_1 \amalg X_2 \xrightarrow{f_1 \sqcup f_2} S] = [X_1 \xrightarrow{f_1} S] + [X_2 \xrightarrow{f_2} S],$$

(b) for every commutative diagram



with g a geometric bijection, a relation

$$[X_1 \xrightarrow{f_1} S] = [X_2 \xrightarrow{f_2} S],$$

(c) for every pair of Zariski fibrations

$$h_1 : X_1 \rightarrow Y, \quad h_2 : X_2 \rightarrow Y$$

with the same fibres, and every morphism $g : Y \rightarrow S$, a relation

$$[X_1 \xrightarrow{g \circ h_1} S] = [X_2 \xrightarrow{g \circ h_2} S].$$

The group $K(\text{St}/S)$ has the structure of a $K(\text{St}/\mathbb{C})$ -module, defined by setting

$$[X] \cdot [Y \xrightarrow{f} S] = [X \times Y \xrightarrow{f \circ \pi_2} S]$$

and extending linearly.

Remark 3.11. Suppose that Λ is a \mathbb{Q} -algebra and

$$\mathcal{Y} : K(\text{St}/\mathbb{C}) \longrightarrow \Lambda$$

is a ring homomorphism. Then for each stack S with affine stabilisers Joyce defines [11, Section 4.3] a Λ -module $\text{SF}(S, \mathcal{Y}, \Lambda)$ whose elements he calls stack functions. It is easy to see that there is an isomorphism of Λ -modules

$$\text{SF}(S, \mathcal{Y}, \Lambda) \cong K(\text{St}/S) \otimes_{K(\text{St}/\mathbb{C})} \Lambda.$$

We leave the proof to the reader.

3.5. Functoriality

The following statements are all easy consequences of the basic properties of fibre products of stacks, and we leave the proofs to the reader. We assume that all stacks appearing have affine stabilisers.

(a) A morphism of stacks $a : S \rightarrow T$ induces a map of $K(\text{St}/\mathbb{C})$ -modules

$$a_* : K(\text{St}/S) \longrightarrow K(\text{St}/T)$$

sending $[X \xrightarrow{f} S]$ to $[X \xrightarrow{a \circ f} T]$.

(b) A morphism of stacks $a : S \rightarrow T$ of finite type induces a map of $K(\text{St}/\mathbb{C})$ -modules

$$a^* : K(\text{St}/T) \longrightarrow K(\text{St}/S)$$

sending $[Y \xrightarrow{g} T]$ to $[X \xrightarrow{f} S]$ where f is the map appearing in the Cartesian square

$$\begin{array}{ccc} X & \longrightarrow & Y \\ f \downarrow & & \downarrow g \\ S & \xrightarrow{a} & T \end{array}$$

(c) The above assignments are functorial, in that

$$(b \circ a)_* = b_* \circ a_*, \quad (b \circ a)^* = a^* \circ b^*,$$

whenever a and b are composable morphisms of stacks with the required properties.

(d) Given a Cartesian square of maps

$$\begin{array}{ccc} U & \xrightarrow{c} & V \\ d \downarrow & & \downarrow b \\ S & \xrightarrow{a} & T \end{array}$$

one has the base-change property

$$b^* \circ a_* = c_* \circ d^* : K(\text{St}/S) \longrightarrow K(\text{St}/V).$$

(e) For every pair of stacks (S_1, S_2) there is a Künneth map

$$K : K(\text{St}/S_1) \otimes K(\text{St}/S_2) \rightarrow K(\text{St}/S_1 \times S_2)$$

given by

$$[X_1 \xrightarrow{f_1} S_1] \otimes [X_2 \xrightarrow{f_2} S_2] \mapsto [X_1 \times X_2 \xrightarrow{f_1 \times f_2} S_1 \times S_2].$$

It is a morphism of $K(\text{St}/\mathbb{C})$ -modules.

We can view the functor $K(\text{St}/-)$ as defining a primitive cohomology theory for stacks.

4. Motivic Hall algebra

Let M be a smooth projective variety and

$$\mathcal{A} = \text{Coh}(M)$$

its category of coherent sheaves. In this section, following Joyce [11], we introduce the motivic Hall algebra of the category \mathcal{A} . Much of what we do here would apply with minor modifications to other abelian categories, but we make no attempt at maximal generality.

We use the following abuse of notation throughout: if $f : T \rightarrow S$ is a morphism of schemes, and E is a sheaf on $S \times M$, we use the shorthand $f^*(E)$ for the pullback to $T \times M$, rather than the more correct $(f \times 1_M)^*(E)$.

4.1. Stacks of flags

Let $\mathcal{M}^{(n)}$ denote the moduli stack of n -flags of coherent sheaves on M . The objects over a scheme S are chains of monomorphisms of coherent sheaves on $S \times M$ of the form

$$0 = E_0 \hookrightarrow E_1 \hookrightarrow \dots \hookrightarrow E_n = E \tag{7}$$

such that each factor $F_i = E_i/E_{i-1}$ is S -flat. It follows that each sheaf E_i is also S -flat. If

$$0 = E'_0 \hookrightarrow E'_1 \hookrightarrow \dots \hookrightarrow E'_n = E$$

is another such object over a scheme T , then a morphism in $\mathcal{M}^{(n)}$ lying over a morphism of schemes $f : T \rightarrow S$ is a collection of isomorphisms of sheaves

$$\theta_i : f^*(E_i) \rightarrow E'_i$$

such that each diagram

$$\begin{array}{ccc}
 f^*(E_i) & \longrightarrow & f^*(E_{i+1}) \\
 \theta_i \downarrow & & \downarrow \theta_{i+1} \\
 E'_i & \longrightarrow & E'_{i+1}
 \end{array}$$

commutes. Here we have taken the usual step of choosing, for each morphism of sheaves, a pullback of every coherent sheaf on its target. The stack property for $\mathcal{M}^{(n)}$ follows easily from the corresponding property of the stack $\mathcal{M} = \mathcal{M}^{(1)}$.

There are morphisms of stacks

$$a_i : \mathcal{M}^{(n)} \rightarrow \mathcal{M}, \quad 1 \leq i \leq n,$$

sending a flag (7) to its i th factor $F_i = E_i/E_{i-1}$. To define these it is first necessary to choose a cokernel for each monomorphism $E_{i-1} \rightarrow E_i$. There is another morphism

$$b : \mathcal{M}^{(n)} \rightarrow \mathcal{M}$$

sending a flag (7) to the sheaf $E_n = E$. Note that the functors defining all these morphisms of stacks have the iso-fibration property of Lemma A.1. Using this it is immediate that for $n > 1$ there is a Cartesian square

$$\begin{array}{ccc}
 \mathcal{M}^{(n)} & \xrightarrow{t} & \mathcal{M}^{(2)} \\
 s \downarrow & & \downarrow a_1 \\
 \mathcal{M}^{(n-1)} & \xrightarrow{b} & \mathcal{M}
 \end{array} \tag{8}$$

where s and t send a flag (7) to the flags

$$E_1 \hookrightarrow \dots \hookrightarrow E_{n-1} \quad \text{and} \quad E_{n-1} \hookrightarrow E_n$$

respectively.

There is a kind of duality around here, which is the basic reason for the associativity of the Hall algebra. Instead of considering flags of the form (7) we could instead consider flags

$$E = E^0 \twoheadrightarrow E^1 \twoheadrightarrow \dots \twoheadrightarrow E^{n-1} \twoheadrightarrow E^n = 0. \tag{9}$$

Setting $E^i = E/E_i$ shows that this gives an isomorphic stack. This dual approach leads to Cartesian diagrams

$$\begin{array}{ccc}
 \mathcal{M}^{(n)} & \xrightarrow{v} & \mathcal{M}^{(2)} \\
 u \downarrow & & \downarrow a_2 \\
 \mathcal{M}^{(n-1)} & \xrightarrow{b} & \mathcal{M}
 \end{array} \tag{10}$$

where u and v send a flag (9) to the flags

$$E^1 \twoheadrightarrow \dots \twoheadrightarrow E^{n-1} \quad \text{and} \quad E^0 \twoheadrightarrow E^1$$

respectively.

The stack $\mathcal{M}^{(2)}$ can be thought of as the stack of short exact sequences in \mathcal{A} . There is a diagram

$$\begin{array}{ccc} \mathcal{M}^{(2)} & \xrightarrow{b} & \mathcal{M} \\ (a_1, a_2) \downarrow & & \\ \mathcal{M} \times \mathcal{M} & & \end{array} \tag{11}$$

where, as above, the morphisms a_1, a_2 and b send a short exact sequence

$$0 \longrightarrow A_1 \longrightarrow B \longrightarrow A_2 \longrightarrow 0$$

to the sheaves A_1, A_2 and B respectively.

Lemma 4.1. *The stacks $\mathcal{M}^{(n)}$ are algebraic.*

Proof. Suppose $f : S \rightarrow \mathcal{M}$ is a morphism of stacks corresponding to a flat family of sheaves B on $S \times M$. Forming the Cartesian square

$$\begin{array}{ccc} Z & \longrightarrow & \mathcal{M}^{(2)} \\ \downarrow & & \downarrow b \\ S & \xrightarrow{f} & \mathcal{M} \end{array}$$

it is easy to see that Z is represented by the relative Quot scheme parameterising quotients of B over S . Thus b is representable, and pulling back an atlas for \mathcal{M} gives an atlas for $\mathcal{M}^{(2)}$. Since fibre products of algebraic stacks are algebraic the result then follows by induction and the existence of the squares (8). \square

The morphism (a_1, a_2) is not representable. The fibre over a point of $\mathcal{M} \times \mathcal{M}$ corresponding to a pair of sheaves (A_1, A_2) is the quotient stack

$$[\text{Ext}^1(A_2, A_1)/\text{Hom}(A_2, A_1)],$$

with the action of the vector space $\text{Hom}_{\mathcal{A}}(A_2, A_1)$ being the trivial one. This statement follows from Proposition 6.2 below.

Lemma 4.2. *The morphism (a_1, a_2) is of finite type.*

Proof. Fix a projective embedding of M . For each integer m and each polynomial P there is a finite type open substack $\mathcal{M}_m(P) \subset \mathcal{M}$ parameterising m -regular sheaves with Hilbert polynomial P . Define an open substack

$$Y = (a_1, a_2)^{-1}(\mathcal{M}_m(P_1) \times \mathcal{M}_m(P_2)) \subset \mathcal{M}^{(2)}.$$

It will be enough to show that Y is of finite type. But the morphism b restricts to a map

$$b : Y \rightarrow \mathcal{M}_m(P_1 + P_2)$$

since an extension of two m -regular sheaves is also m -regular. This map is of finite type, because once one fixes the Hilbert polynomials involved, the relative Quot scheme is of finite type. \square

4.2. The Hall algebra

Let us set

$$H(\mathcal{A}) = K(\text{St}/\mathcal{M}).$$

Applying the results of Section 3.5 to the diagram (11) gives a morphism of $K(\text{St}/\mathbb{C})$ -modules

$$m = b_* \circ (a_1, a_2)^* : H(\mathcal{A}) \otimes H(\mathcal{A}) \longrightarrow H(\mathcal{A})$$

which we call the convolution product.² Explicitly this is given by the rule

$$[X_1 \xrightarrow{f_1} \mathcal{M}] * [X_2 \xrightarrow{f_2} \mathcal{M}] = [Z \xrightarrow{boh} \mathcal{M}],$$

where h is defined by the following Cartesian square

$$\begin{array}{ccccc} Z & \xrightarrow{h} & \mathcal{M}^{(2)} & \xrightarrow{b} & \mathcal{M} \\ \downarrow & & \downarrow (a_1, a_2) & & \\ X_1 \times X_2 & \xrightarrow{f_1 \times f_2} & \mathcal{M} \times \mathcal{M} & & \end{array}$$

The following result is due to Joyce [11, Theorem 5.2], although the basic idea is of course the same as for previous incarnations of the Hall algebra.

Theorem 4.3. *The product m gives $H(\mathcal{A})$ the structure of an associative unital algebra over $K(\text{St}/\mathbb{C})$. The unit element is*

$$1 = [\mathcal{M}_0 \subset \mathcal{M}],$$

where $\mathcal{M}_0 \cong \text{Spec}(\mathbb{C})$ is the substack of zero objects in \mathcal{A} .

² Here and in the rest of this section we suppress an application of the Künneth map

$$K(\text{St}/\mathcal{M}) \otimes K(\text{St}/\mathcal{M}) \rightarrow K(\text{St}/\mathcal{M} \times \mathcal{M})$$

from the notation.

Proof. Consider the composition

$$H(\mathcal{A}) \otimes H(\mathcal{A}) \otimes H(\mathcal{A}) \xrightarrow{m \otimes \text{id}} H(\mathcal{A}) \otimes H(\mathcal{A}) \xrightarrow{m} H(\mathcal{A}).$$

It is induced by the diagram

$$\begin{array}{ccc} & \mathcal{M}^{(2)} & \xrightarrow{b} \mathcal{M} \\ & (a_1, a_2) \downarrow & \\ \mathcal{M}^{(2)} \times \mathcal{M} & \xrightarrow{(b, \text{id})} & \mathcal{M} \times \mathcal{M} \\ (a_1, a_2, \text{id}) \downarrow & & \\ \mathcal{M} \times \mathcal{M} \times \mathcal{M} & & \end{array}$$

There is a bigger commutative diagram obtained by filling in the top left square with

$$\begin{array}{ccc} \mathcal{M}^{(3)} & \xrightarrow{t} & \mathcal{M}^{(2)} \\ (s, a_2 \circ t) \downarrow & & \downarrow (a_1, a_2) \\ \mathcal{M}^{(2)} \times \mathcal{M} & \xrightarrow{(b, \text{id})} & \mathcal{M} \times \mathcal{M} \end{array}$$

where s sends a flag $E_1 \hookrightarrow E_2 \hookrightarrow E_3$ to the flag $E_1 \hookrightarrow E_2$, and t sends it to $E_2 \hookrightarrow E_3$. This square is Cartesian because of the square (8) and Lemma A.3, so by the base-change property of Section 3.5

$$m \circ (m \otimes \text{id}) = b_* \circ (a_1, a_2, a_3)^*$$

is induced by the diagram

$$\begin{array}{ccc} \mathcal{M}^{(3)} & \xrightarrow{b} & \mathcal{M} \\ (a_1, a_2, a_3) \downarrow & & \\ \mathcal{M}^3 & & \end{array}$$

A similar argument using the square (10) shows that the other composition is induced by the same diagram. The multiplication is therefore associative. We leave the reader to check the unit property. \square

Lemma 4.4. *The n -fold product*

$$m_n : H(\mathcal{A})^{\otimes n} \rightarrow H(\mathcal{A})$$

is induced by the diagram

$$\begin{array}{ccc} \mathcal{M}^{(n)} & \xrightarrow{b} & \mathcal{M} \\ (a_1, \dots, a_n) \downarrow & & \\ \mathcal{M}^n & & \end{array}$$

in the sense that

$$m_n = b_* \circ (a_1, \dots, a_n)^* : H(\mathcal{A})^{\otimes n} \longrightarrow H(\mathcal{A}).$$

Proof. This follows by induction and a similar argument to the one above, but using the Cartesian diagram

$$\begin{array}{ccc} \mathcal{M}^{(n)} & \xrightarrow{t} & \mathcal{M}^{(2)} \\ (s, a_2 \circ t) \downarrow & & \downarrow (a_1, a_2) \\ \mathcal{M}^{(n-1)} \times \mathcal{M} & \xrightarrow{(b, \text{id})} & \mathcal{M} \times \mathcal{M} \end{array}$$

given by (8) and Lemma A.3. \square

4.3. Grading

Let $K(M) = K(\mathcal{A})$ denote the Grothendieck group of the category \mathcal{A} . Given two coherent sheaves E and F we can define

$$\chi(E, F) = \sum_i (-1)^i \dim_{\mathbb{C}} \text{Ext}^i(E, F).$$

This defines a bilinear form $\chi(-, -)$ on $K(M)$ called the Euler form. Serre duality implies that the left and right radicals ${}^\perp K(M)$ and $K(M)^\perp$ are equal. The numerical Grothendieck group is the quotient

$$N(M) = K(M)/K(M)^\perp.$$

Let $\Gamma \subset N(M)$ denote the monoid of effective classes, which is to say classes of the form $[E]$ with E a sheaf.

Lemma 4.5. *Suppose S is a connected scheme and F is an S -flat coherent sheaf on $S \times M$. For each point $s \in S(\mathbb{C})$ let*

$$F_s = F|_{\{s\} \times M}$$

be the corresponding sheaf on M . Then the class $[F_s] \in N(M)$ is independent of the point s .

Proof. For any locally-free sheaf E , the integer

$$\chi(E, F_s) = \chi(E^\vee \otimes F_s)$$

is locally constant on S . Since M is smooth, the Grothendieck group $K(M)$ is spanned by the classes of locally-free sheaves, so the class $[F_s] \in N(M)$ is also locally constant. \square

It follows from Lemma 4.5 that the stack \mathcal{M} splits as a disjoint union of open and closed substacks

$$\mathcal{M} = \bigsqcup_{\alpha \in \Gamma} \mathcal{M}_\alpha,$$

where $\mathcal{M}_\alpha \subset \mathcal{M}$ is the stack of objects of class $\alpha \in \Gamma$. The inclusion $\mathcal{M}_\alpha \subset \mathcal{M}$ induces an embedding

$$K(\text{St}/\mathcal{M}_\alpha) \subset K(\text{St}/\mathcal{M}).$$

There is thus a direct sum decomposition

$$H(\mathcal{A}) = \bigoplus_{\alpha \in \Gamma} H(\mathcal{A})_\alpha, \tag{12}$$

and $H(\mathcal{A})$ with the convolution product becomes a Γ -graded algebra.

4.4. Sheaves supported in dimension $\leq d$

For any integer d there is a full abelian subcategory

$$\mathcal{A}_{\leq d} = \text{Coh}_{\leq d}(M) \subset \mathcal{A} = \text{Coh}(M)$$

closed under extensions, consisting of coherent sheaves on M whose support has dimension $\leq d$. There is a subgroup

$$N_{\leq d}(M) \subset N(M)$$

spanned by classes of objects of $\text{Coh}_{\leq d}(M)$, and a corresponding positive cone

$$\Gamma_{\leq d} = N_{\leq d}(M) \cap \Gamma.$$

One can define a Hall algebra $H(\mathcal{A}_{\leq d})$ by replacing the moduli stack \mathcal{M} in the above discussion with the substack $\mathcal{M}_{\leq d}$ of objects of $\mathcal{A}_{\leq d}$.

In fact it is easy to see that a coherent sheaf E lies in $\text{Coh}_{\leq d}(M)$ precisely if its class $[E] \in N(M)$ lies in the subgroup $N_{\leq d}(M)$. Thus

$$\mathcal{M}_{\leq d} = \bigsqcup_{\alpha \in \Gamma_{\leq d}} \mathcal{M}_\alpha,$$

and there is an identification

$$H(\mathcal{A}_{\leq d}) = \bigoplus_{\alpha \in \Gamma_{\leq d}} H_{\alpha}(\mathcal{A}).$$

We will make heavy use of the $d = 1$ version of this construction in [3].

5. Integration map

In this section we construct a homomorphism of Poisson algebras from a semi-classical limit of the Hall algebra to an algebra of functions on a symplectic torus. It can be viewed as the semi-classical limit of the ring homomorphism envisaged by Kontsevich and Soibelman [14]. We assume throughout³ that M is a smooth projective Calabi–Yau threefold over \mathbb{C} . We include in this the condition that

$$H^1(M, \mathcal{O}_M) = 0.$$

There are two versions of the story depending on a choice of sign $\sigma \in \{\pm 1\}$ which we fix throughout. The choice $\sigma = +1$ will lead to naive Euler characteristic invariants, while $\sigma = -1$ leads to Donaldson–Thomas invariants.

5.1. Regular elements

Consider the maps of commutative rings

$$K(\text{Var}/\mathbb{C}) \rightarrow K(\text{Var}/\mathbb{C})[[\mathbb{L}^{-1}]] \rightarrow K(\text{St}/\mathbb{C}),$$

and recall that $H(\mathcal{A})$ is an algebra over $K(\text{St}/\mathbb{C})$. Define a $K(\text{Var}/\mathbb{C})[[\mathbb{L}^{-1}]]$ -module

$$H_{\text{reg}}(\mathcal{A}) \subset H(\mathcal{A})$$

to be the span of classes of maps $[X \xrightarrow{f} \mathcal{M}]$ with X a variety. We call an element of $H(\mathcal{A})$ regular if it lies in this submodule. The following result will be proved in Section 7.1 below.

Theorem 5.1. *The submodule of regular elements is closed under the convolution product:*

$$H_{\text{reg}}(\mathcal{A}) * H_{\text{reg}}(\mathcal{A}) \subset H_{\text{reg}}(\mathcal{A}),$$

and is therefore a $K(\text{Var}/\mathbb{C})[[\mathbb{L}^{-1}]]$ -algebra. Moreover the quotient

$$H_{\text{sc}}(\mathcal{A}) = H_{\text{reg}}(\mathcal{A})/(\mathbb{L} - 1)H_{\text{reg}}(\mathcal{A})$$

is a commutative $K(\text{Var}/\mathbb{C})$ -algebra.

³ In fact Theorem 5.1 and the first part of Theorem 5.2 (the existence of the algebra map) hold for an arbitrary smooth projective variety M .

We call the algebra $H_{sc}(\mathcal{A})$ the semi-classical Hall algebra. Since

$$[\mathbb{C}^*] = \mathbb{L} - 1$$

is invertible in $K(\text{St}/\mathbb{C})$, there is a Poisson bracket on $H(\mathcal{A})$ given by the formula

$$\{f, g\} = \frac{f * g - g * f}{\mathbb{L} - 1}.$$

This bracket preserves the subalgebra $H_{reg}(\mathcal{A})$ because the multiplication in $H_{reg}(\mathcal{A})$ is commutative modulo the ideal $(\mathbb{L} - 1)$. The induced bracket on $H_{reg}(\mathcal{A})$ then descends to give a Poisson bracket on the commutative algebra $H_{sc}(\mathcal{A})$.

5.2. The integration map

Define a ring

$$\mathbb{Z}_\sigma[\Gamma] = \bigoplus_{\alpha \in \Gamma} \mathbb{Z} \cdot x^\alpha$$

by taking the free abelian group spanned by symbols x^α for $\alpha \in \Gamma$ and setting

$$x^\alpha * x^\beta = \sigma^{\chi(\alpha, \beta)} \cdot x^{\alpha + \beta}.$$

Since the Euler form is skew-symmetric this ring is commutative. Equip it with a Poisson structure by defining

$$\{x^\alpha, x^\beta\} = \sigma^{\chi(\alpha, \beta)} \cdot \chi(\alpha, \beta) \cdot x^{\alpha + \beta}.$$

Take a locally constructible function

$$\lambda : \mathcal{M} \rightarrow \mathbb{Z}.$$

For definitions of constructible functions on stacks see [8]. Associated to every sheaf $E \in \mathcal{A}$ is an integral weight

$$\lambda(E) \in \mathbb{Z}.$$

If X is a variety with a map $f : X \rightarrow \mathcal{M}$, there is an induced constructible function

$$f^*(\lambda) : X \rightarrow \mathbb{Z},$$

and a weighted Euler characteristic

$$\chi(X, f^*(\lambda)) = \sum_{n \in \mathbb{Z}} n \cdot \chi((\lambda \circ f)^{-1}(n)).$$

We prove the following result in Section 7.2 below.

Theorem 5.2. *Given a locally constructible function $\lambda : \mathcal{M} \rightarrow \mathbb{Z}$, there is a unique group homomorphism*

$$I : H_{sc}(\mathcal{A}) \longrightarrow \mathbb{Z}_\sigma[\Gamma],$$

such that if X is a variety and $f : X \rightarrow \mathcal{M}$ factors via $\mathcal{M}_\alpha \subset \mathcal{M}$ then

$$I([X \xrightarrow{f} \mathcal{M}]) = \chi(X, f^*(\lambda)) \cdot x^\alpha.$$

Moreover, I is a homomorphism of commutative algebras if for all $E, F \in \mathcal{A}$

$$\lambda(E \oplus F) = \sigma^{\chi(E, F)} \cdot \lambda(E) \cdot \lambda(F), \tag{13}$$

and a homomorphism of Poisson algebras if, in addition, the expression

$$m(E, F) = \chi(\mathbb{P}\text{Ext}^1(F, E), \lambda(G_\theta) - \lambda(G_0)) \tag{14}$$

is symmetric in E and F .

This statement may need a little explanation. We have written G_θ for the sheaf

$$0 \longrightarrow E \longrightarrow G_\theta \longrightarrow F \longrightarrow 0$$

corresponding to a class $\theta \in \text{Ext}^1(F, E)$. In particular $G_0 = E \oplus F$. For non-zero θ the isomorphism class of the object G_θ depends only on the class

$$[\theta] \in \mathbb{P}\text{Ext}^1(F, E),$$

and it is easy to see that the map $[\theta] \mapsto \lambda(G_\theta)$ is a constructible function on the projective space $\mathbb{P}\text{Ext}^1(F, E)$.

5.3. Behrend function

Recall [1] that Behrend associates to any scheme S of finite type over \mathbb{C} a constructible function

$$\nu_S : S \rightarrow \mathbb{Z}$$

with the property that if S is a proper moduli scheme with a symmetric obstruction theory then the associated Donaldson–Thomas virtual count coincides with the weighted Euler characteristic:

$$\#_{\text{vir}}(S) := \int_{[S]^{\text{vir}}} 1 = \chi(S, \nu_S).$$

These functions satisfy the relation

$$f^*(v_S) = (-1)^d v_T \tag{15}$$

whenever $f : T \rightarrow S$ is a smooth morphism of relative dimension d . It follows easily from this that every stack S , locally of finite type over \mathbb{C} , has an associated locally constructible function

$$v_S : S \rightarrow \mathbb{Z}$$

defined uniquely by the condition that (15) also holds for smooth morphisms of stacks.

It is clear that the constant function

$$\mathbf{1} : \mathcal{M} \rightarrow \mathbb{Z}$$

satisfies the conditions of Theorem 5.2 with the sign $\sigma = +1$. Regarding the Behrend function, Joyce and Song [12, Theorem 5.9] proved the following wonderful result. For their proof (which uses gauge-theoretic methods) it is essential that our base variety M is proper and that we are working over \mathbb{C} .

Theorem 5.3. *The Behrend function*

$$v_{\mathcal{M}} : \mathcal{M} \rightarrow \mathbb{Z}$$

for the moduli stack \mathcal{M} satisfies the conditions of Theorem 5.2 with the sign $\sigma = -1$.

Thus we have (at least) two integration maps: taking $\lambda = \mathbf{1}$ leads to invariants defined by unweighted Euler characteristics, whereas taking $\lambda = v_{\mathcal{M}}$ leads to Donaldson–Thomas invariants.

6. Fibres of the convolution map

In this section we give a careful analysis of the fibres of the morphism of stacks (a_1, a_2) appearing in the definition of the convolution product. This will be the main tool in the proofs of Theorems 5.1 and 5.2 we give in the next section. We again adopt the abuse of notation for pullbacks of families of sheaves explained in the preamble to Section 4.

6.1. Universal extensions

Suppose S is an affine scheme and E_1 and E_2 are coherent sheaves on $S \times M$, flat over S . Let Aff/S denote the category of affine schemes over S . Define a functor

$$\Phi_S^k(E_1, E_2) : \text{Aff}/S \rightarrow \text{Ab}$$

by sending an object $f : T \rightarrow S$ to the abelian group

$$\Phi_S^k(E_1, E_2)(f) = \text{Ext}_{T \times M}^k(f^*(E_1), f^*(E_2)).$$

The image of a morphism

$$\begin{array}{ccc}
 U & \xrightarrow{h} & T \\
 \searrow g & & \swarrow f \\
 & & S
 \end{array}$$

in Aff/S is defined using the canonical map

$$h^* : \text{Ext}_{T \times M}^k(f^*(E_1), f^*(E_2)) \rightarrow \text{Ext}_{U \times M}^k(h^*f^*(E_1), h^*f^*(E_2))$$

together with the canonical isomorphisms

$$\text{can} : g^*(E_i) \cong h^*f^*(E_i).$$

To check that this does indeed define a functor one needs to apply the uniqueness properties of pullback in the usual way (see for example [23, Section 3.2.1]).

Consider the object

$$\mathbf{R}\mathcal{H}\text{om}_{\mathcal{O}_S}(E_1, E_2) = \mathbf{R}\pi_{S,*}\mathbf{R}\mathcal{H}\text{om}_{\mathcal{O}_{S \times M}}(E_1, E_2) \in \text{DCoh}(S).$$

For each $k \geq 0$ we set

$$\text{Ext}_{\mathcal{O}_S}^k(E_1, E_2) := H^k(\mathbf{R}\mathcal{H}\text{om}_{\mathcal{O}_S}(E_1, E_2)) \in \text{Coh}(S).$$

We shall say that E_1 and E_2 have constant Ext groups if these sheaves are all locally-free.

Lemma 6.1. *Suppose E_1 and E_2 are S -flat coherent sheaves on $S \times M$ with constant Ext groups. Then the functor*

$$\Phi_S^k(E_1, E_2) : \text{Aff}/S \rightarrow \text{Ab}$$

defined above is represented by the vector bundle $V^k(S)$ over S corresponding to the locally-free sheaf $\text{Ext}_{\mathcal{O}_S}^k(E_1, E_2)$.

Proof. If $f : T \rightarrow S$ is a morphism of schemes then by flat base-change,

$$\mathbf{R}\mathcal{H}\text{om}_{\mathcal{O}_T}(f^*(E_1), f^*(E_2)) \cong \mathbf{L}f^* \circ \mathbf{R}\mathcal{H}\text{om}_{\mathcal{O}_S}(E_1, E_2).$$

If E_1 and E_2 have constant Ext groups it follows that

$$\text{Ext}_{\mathcal{O}_T}^k(f^*(E_1), f^*(E_2)) \cong f^*\text{Ext}_{\mathcal{O}_S}^k(E_1, E_2).$$

There is also an identity

$$\mathbf{R}\mathcal{H}\text{om}_{T \times M}(f^*(E_1), f^*(E_2)) \cong \mathbf{R}\Gamma \circ \mathbf{R}\mathcal{H}\text{om}_{\mathcal{O}_T}(f^*(E_1), f^*(E_2)).$$

If T is affine it follows that

$$\Phi_S^k(E_1, E_2)(f) \cong \Gamma(T, f^*(\text{Ext}_{\mathcal{O}_S}^k(E_1, E_2))) \cong \text{Map}_S(T, V^k(S)).$$

These isomorphisms commute with pullback and hence define an isomorphism of functors. \square

6.2. The convolution morphism

The main result of this section is as follows.

Proposition 6.2. *Let X_1 and X_2 be varieties with morphisms*

$$f_1 : X_1 \rightarrow \mathcal{M}, \quad f_2 : X_2 \rightarrow \mathcal{M},$$

and let $E_i \in \text{Coh}(X_i \times M)$ be the corresponding families of sheaves on M . Then we can stratify $X_1 \times X_2$ by locally-closed affine subvarieties

$$S \subset X_1 \times X_2$$

with the following property. For each point $s \in S(\mathbb{C})$ the space

$$V^k(s) = \text{Ext}_M^k(E_2|_{\{s\} \times M}, E_1|_{\{s\} \times M})$$

has a fixed dimension $d_k(S)$, and if we form the Cartesian squares

$$\begin{array}{ccccc} Z_S & \longrightarrow & Z & \xrightarrow{h} & \mathcal{M}^{(2)} \\ u \downarrow & & t \downarrow & & \downarrow (a_1, a_2) \\ S & \longrightarrow & X_1 \times X_2 & \xrightarrow{f_1 \times f_2} & \mathcal{M} \times \mathcal{M} \end{array}$$

then

$$Z_S \cong S \times [\mathbb{C}^{d_1(S)} / \mathbb{C}^{d_0(S)}],$$

where the vector space $\mathbb{C}^{d_0(S)}$ acts trivially, and the map u is the obvious projection.

Proof. By the existence of flattening stratifications, we can stratify $X_1 \times X_2$ by locally-closed subschemes S such that the restrictions of the families E_1 and E_2 have constant Ext groups. To ease the notation set $A = E_1$, $B = E_2$ and

$$V^k(S) = \text{Ext}_{\mathcal{O}_S}^k(B, A)$$

considered as a vector bundle over S . The projection morphism

$$p : V^k(S) \rightarrow S$$

defines an abelian group scheme over S . We will show that

$$Z_S \cong [V^1(S)/V^0(S)], \tag{16}$$

where the action of $V^0(S)$ on $V^1(S)$ is the trivial one. This will be enough because refining the stratification if necessary we can assume that each of the bundles $V^i(S)$ is trivial with fibre $\mathbb{C}^{d_i(S)}$. Then there are obvious isomorphisms

$$[V^1(S)/V^0(S)] \cong [S \times \mathbb{C}^{d_1(S)}/S \times \mathbb{C}^{d_0(S)}] \cong S \times [\mathbb{C}^{d_1(S)}/\mathbb{C}^{d_0(S)}].$$

Using Lemma A.1 one derives the following description of the stack Z_S . The objects over a scheme T consist of a morphism $f : T \rightarrow S$ and a short exact sequence of T -flat sheaves on $T \times M$ of the form

$$0 \longrightarrow f^*(A) \xrightarrow{\alpha} E \xrightarrow{\beta} f^*(B) \longrightarrow 0. \tag{17}$$

Suppose given another such map $g : U \rightarrow S$ and a sequence

$$0 \longrightarrow g^*(A) \xrightarrow{\gamma} F \xrightarrow{\delta} g^*(B) \longrightarrow 0.$$

Then a morphism between these two objects in Z_S is a commuting diagram of schemes

$$\begin{array}{ccc} U & \xrightarrow{h} & T \\ & \searrow g & \swarrow f \\ & & S \end{array}$$

and an isomorphism of sheaves $\theta : h^*(E) \rightarrow F$ such that the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & h^*f^*(A) & \xrightarrow{h^*(\alpha)} & h^*(E) & \xrightarrow{h^*(\beta)} & h^*f^*(B) \longrightarrow 0 \\ & & \text{can} \downarrow & & \theta \downarrow & & \text{can} \downarrow \\ 0 & \longrightarrow & g^*(A) & \xrightarrow{\gamma} & F & \xrightarrow{\delta} & g^*(B) \longrightarrow 0 \end{array} \tag{18}$$

commutes. Here can denotes the canonical isomorphism.

Lemma 6.1 implies that there is a universal extension class

$$\eta \in \text{Ext}_{V^1(S) \times M}^1(p^*(B), p^*(A)).$$

Choose a corresponding short exact sequence

$$0 \longrightarrow p^*(A) \xrightarrow{\gamma} F \xrightarrow{\delta} p^*(B) \longrightarrow 0. \tag{19}$$

This defines an object of $Z_S(V^1(S))$ and hence a morphism of stacks

$$q : V^1(S) \rightarrow Z_S.$$

Consider the fibre product

$$W = V^1(S) \times_{Z_S} V^1(S).$$

For each scheme T the groupoid $W(T)$ is a set, so we can identify W with the corresponding functor

$$W : \text{Sch}/\mathbb{C} \rightarrow \text{Set}.$$

It will be enough to show that the functor W is isomorphic to the functor

$$\Phi_{V^1(S)}^0(p^*(B), p^*(A))$$

and that there is a Cartesian diagram of stacks

$$\begin{array}{ccc} V^1(S) \times_S V^0(S) & \xrightarrow{\pi_1} & V^1(S) \\ \pi_1 \downarrow & & \downarrow q \\ V^1(S) & \xrightarrow{q} & Z_S \end{array}$$

Then Z_S is isomorphic to the quotient stack corresponding to the trivial action of $V^0(S)$ on $V^1(S)$ as claimed.

By the universal property of $V^1(S)$ a morphism $a : T \rightarrow V^1(S)$ corresponds to a morphism $f : T \rightarrow S$ together with an extension class

$$\zeta \in \text{Ext}_{S \times M}^1(f^*(B), f^*(A)).$$

Under this correspondence $f = p \circ a$. The composite morphism

$$q \circ a : T \rightarrow Z_S$$

then corresponds to the object of $Z_S(T)$ defined by the morphism f and the short exact sequence

$$0 \longrightarrow f^*(A) \longrightarrow a^*(F) \longrightarrow f^*(B) \longrightarrow 0$$

obtained by applying a^* to (19) and composing with the canonical isomorphisms.

Suppose $b : T \rightarrow V^1(S)$ is another morphism corresponding to a morphism $g : T \rightarrow S$ and an extension class

$$\eta \in \text{Ext}_{S \times M}^1(g^*(B), g^*(A)).$$

Then there is an isomorphism of the corresponding objects of $Z_S(T)$ lying over the identity of T precisely if $f = g$ and there is an isomorphism of short exact sequences

$$\begin{array}{ccccccc}
 0 & \longrightarrow & f^*(A) & \longrightarrow & a^*(F) & \longrightarrow & f^*(B) \longrightarrow 0 \\
 & & \downarrow = & & \downarrow & & \downarrow = \\
 0 & \longrightarrow & f^*(A) & \longrightarrow & b^*(F) & \longrightarrow & f^*(B) \longrightarrow 0
 \end{array} \tag{20}$$

In particular, it follows that $\zeta = \eta$, and hence by the universal property of $V^1(S)$ one has $a = b$. Moreover the set of possible isomorphisms is in bijection with

$$\text{Hom}_{T \times M}(f^*(B), f^*(A)).$$

Thus the elements of the set $W(T)$ consist of a morphism $a : T \rightarrow V^1(S)$ and an element of

$$\Phi_{V^1(S)}^0(p^*(B), p^*(A))(a).$$

We leave it to the reader to check that this correspondence commutes with pullback and hence defines an isomorphism of functors. \square

7. Proofs of Theorems 5.1 and 5.2

Using Proposition 6.2 we can now give the proofs of Theorems 5.1 and 5.2.

7.1. Proof of Theorem 5.1

Consider two elements

$$a_i = [X_i \xrightarrow{f_i} \mathcal{M}] \in \text{H}_{\text{reg}}(\mathcal{A}), \quad i = 1, 2,$$

with X_1 and X_2 varieties. Let E_i be the family of coherent sheaves on X_i corresponding to the map f_i . Stratify $X_1 \times X_2$ by locally-closed subvarieties S_j as in Proposition 6.2. In particular, the vector spaces

$$V^k(x_1, x_2) = \text{Ext}_M^k(E_2|_{\{x_2\} \times M}, E_1|_{\{x_1\} \times M}), \quad (x_1, x_2) \in S_j,$$

have constant dimension $d_k(S_j)$. Consider the diagram

$$\begin{array}{ccccccc}
 Z_j & \longrightarrow & Z & \xrightarrow{q} & \mathcal{M}^{(2)} & \xrightarrow{h} & \mathcal{M} \\
 t_j \downarrow & & \downarrow \iota & & \downarrow (a_1, a_2) & & \\
 S_j & \longrightarrow & X_1 \times X_2 & \xrightarrow{f_1 \times f_2} & \mathcal{M} \times \mathcal{M} & &
 \end{array}$$

According to Proposition 6.2 one has

$$Z_j \cong [Q_j / \mathbb{C}^{d_0(S_j)}]$$

where $Q_j = V^1(S_j)$ is the total space of a trivial vector bundle over S_j with fibre $V^1(x_1, x_2)$ over a point (x_1, x_2) . Since the Z_j stratify Z it follows that

$$a_1 * a_2 = [Z \xrightarrow{boh} \mathcal{M}] = \sum_j \mathbb{L}^{-d_0(S_j)} [Q_j \xrightarrow{g_j} \mathcal{M}],$$

which is regular. Here the morphism g_j is induced by the universal extension of the families E_1 and E_2 over S_j .

For the second claim split Q_j into the zero-section and its complement. The latter is a \mathbb{C}^* bundle over the associated projective bundle, and it is easy to see that the morphism to \mathcal{M} factors via this map. Thus

$$a_1 * a_2 = \sum_j \mathbb{L}^{-d_0(S_j)} ([S_j \xrightarrow{k} \mathcal{M}] + (\mathbb{L} - 1)[\mathbb{P}(Q_j) \xrightarrow{g_j} \mathcal{M}]), \tag{21}$$

where the morphism k is induced by the direct sum of the families E_1 and E_2 . We therefore obtain

$$a_1 * a_2 = \sum_j [S_j \xrightarrow{k} \mathcal{M}] = [X_1 \times X_2 \xrightarrow{k} \mathcal{M}] \text{ mod } (\mathbb{L} - 1). \tag{22}$$

Clearly we would get the same answer if we calculated $a_2 * a_1$. \square

7.2. Proof of Theorem 5.2

We first check that the map I is well-defined; it is then clearly unique. Stratify \mathcal{M} by locally-closed substacks \mathcal{M}_τ such that λ has constant value $\lambda(\tau)$ on \mathcal{M}_τ . There are projection maps

$$\pi_\tau : K(\text{St}/\mathcal{M}) \rightarrow K(\text{St}/\mathcal{M})$$

defined by taking the fibre product with the inclusion $\mathcal{M}_\tau \subset \mathcal{M}$. For any $a \in K(\text{St}/\mathcal{M})$ there is a canonical decomposition

$$a = \sum_i \pi_\tau(a),$$

where only finitely many of the terms are non-zero. If $a \in K(\text{St}/\mathcal{M})$ is regular so are each of the $\pi_\tau(a)$. On the other hand if $b \in K(\text{St}/\mathcal{M})$ is regular, we can project to an element of $K(\text{St}/\mathbb{C})$, and using Lemmas 2.3 and 3.9 obtain a well-defined Euler characteristic $\chi(b) \in \mathbb{Z}$. Thus we can define a group homomorphism I by the formula

$$I(a) = \sum_i \lambda(\tau) \chi(\pi_\tau(a))$$

and this will clearly have the property stated in the theorem.

Now take notation as in the proof of Theorem 5.1. By Serre duality, we have

$$V^k(x_1, x_2) = V^{3-k}(x_2, x_1)^*. \tag{23}$$

Let $\hat{Q}_j = V^2(S_j)$ be the bundle over S_j whose fibre at (x_1, x_2) is $V^1(x_1, x_2)$. Let

$$g_j : Q_j \rightarrow \mathcal{M}, \quad \hat{g}_j : \hat{Q}_j \rightarrow \mathcal{M},$$

be the morphisms induced by taking the universal extensions of the families E_1 and E_2 over S_j .

We can assume that f_i maps into $\mathcal{M}_{\alpha_i} \subset \mathcal{M}$ and that $f_i^*(\lambda)$ is equal to the constant function with value n_i . Then

$$I(a_i) = n_i \cdot \chi(X_i) \cdot x^{\alpha_i}.$$

Since $\chi(\mathbb{L}) = 1$, the expression (21) shows that

$$I(a_1 * a_2) = \chi(X_1 \times X_2, k^*(\lambda)) \cdot x^{\alpha_1 + \alpha_2}.$$

Using the first assumption (13), we therefore obtain

$$I(a_1 * a_2) = \sigma^{\chi(\alpha_1, \alpha_2)} \cdot n_1 n_2 \cdot \chi(X_1 \times X_2) = I(a_1) * I(a_2).$$

To compute the Poisson bracket we use (21) again. Applying (23) with $k = 0$, and noting that

$$\frac{\mathbb{L}^n - \mathbb{L}^m}{\mathbb{L} - 1} = n - m \pmod{\mathbb{L} - 1},$$

we obtain

$$\{a_1, a_2\} = \sum_j ((d_3(S_j) - d_0(S_j)) \cdot [S_j \xrightarrow{k} \mathcal{M}] + [\mathbb{P}(Q_j) \xrightarrow{g_j} \mathcal{M}] - [\mathbb{P}(\hat{Q}_j) \xrightarrow{\hat{g}_j} \mathcal{M}]).$$

To compute the Euler characteristic of a constructible function over $\mathbb{P}(Q_j)$ it follows from [18, Proposition 1] (see also [22, Corollary 5.1]) that we can first integrate over the fibres of the projection

$$\mathbb{P}(Q_j) \rightarrow S_j$$

and then integrate the resulting constructible function on the base S_j . The second assumption (14) together with $\chi(\mathbb{P}(\mathbb{C}^n)) = n$ therefore gives

$$\chi(\mathbb{P}(Q_j), g_j^*(\lambda)) - \chi(\mathbb{P}(\hat{Q}_j), \hat{g}_j^*(\lambda)) = d_1(S_j) \cdot \chi(S_j, k^*(\lambda)) - d_2(S_j) \cdot \chi(S_j, k^*(\lambda)),$$

and so

$$\begin{aligned} I(\{a_1, a_2\}) &= \chi(\alpha_1, \alpha_2) \cdot \chi(X_1 \times X_2, k^*(\lambda)) \cdot x^{\alpha_1 + \alpha_2} \\ &= \sigma^{\chi(\alpha_1, \alpha_2)} \cdot n_1 n_2 \cdot \chi(\alpha_1, \alpha_2) \cdot \chi(X_1 \times X_2) \cdot x^{\alpha_1 + \alpha_2} = \{I(a_1), I(a_2)\} \end{aligned}$$

as required. \square

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Appendix A. Fibre products of stacks

Here we collect some well-known material on fibre products of stacks.

A.1. Fibre product

Suppose given morphisms of stacks

$$f : X \rightarrow Z, \quad g : Y \rightarrow Z.$$

Recall the definition of the fibre product stack $X \times_Z Y$ and the 2-commutative diagram

$$\begin{array}{ccc} X \times_Z Y & \xrightarrow{\pi_Y} & Y \\ \pi_X \downarrow & & \downarrow g \\ X & \xrightarrow{f} & Z \end{array}$$

The objects of $X \times_Z Y$ are triples (x, y, θ) , where x and y are objects of X and Y over the same scheme S , and $\theta : f(x) \rightarrow g(y)$ is an isomorphism in the groupoid $Z(S)$. A morphism

$$(\alpha, \beta) : (x, y, \theta) \rightarrow (x', y', \theta')$$

consists of morphisms $\alpha : x \rightarrow x'$ in X and $\beta : y \rightarrow y'$ in Y such that the diagram

$$\begin{array}{ccc} f(x) & \xrightarrow{\theta} & g(y) \\ f(\alpha) \downarrow & & \downarrow g(\beta) \\ f(x') & \xrightarrow{\theta'} & g(y') \end{array}$$

commutes. The morphisms π_X and π_Y are defined in the obvious way.

We will call a morphism of stacks $f : X \rightarrow Z$ an iso-fibration if the following property holds. Suppose S is a scheme and

$$\theta : a \rightarrow b$$

is an isomorphism in the groupoid $Z(S)$. Suppose that there is an $a' \in X(S)$ such that $f(a') = a$. Then there is an isomorphism

$$\theta' : a' \rightarrow b'$$

in $X(S)$ such that $f(\theta') = \theta$. The following easy lemma will simplify many computations of such fibre products.

Lemma A.1. *With notation as above, define a full subcategory $W \subset X \times_Z Y$ whose objects are triples (x, y, θ) as above for which there is an object $z \in Z$ with*

$$f(x) = z = g(y), \quad \theta = \text{id}_z.$$

Suppose one of the morphisms f or g is an iso-fibration. Then the inclusion functor $W \rightarrow X \times_Z Y$ is an equivalence of categories.

Proof. For definiteness suppose that it is f that is an iso-fibration. Given an object

$$(x, y, \theta) \in X \times_Z Y$$

take an object $x' \in X$ such that $f(x') = g(y)$ and a morphism $\alpha : x \rightarrow x'$ such that $f(\alpha) = \theta$. Then

$$(\alpha, \text{id}_y) : (x, y, \theta) \rightarrow (x', y, \text{id})$$

defines an isomorphism. Thus (x, y, θ) is isomorphic to an object of the subcategory W . \square

A.2. Cartesian diagrams

A diagram of stacks

$$\begin{array}{ccc} W & \xrightarrow{h} & Y \\ j \downarrow & & \downarrow g \\ X & \xrightarrow{f} & Z \end{array}$$

is called 2-Cartesian if there is an equivalence of stacks

$$t : W \rightarrow X \times_Z Y$$

such that $j = \pi_X \circ t$ and $h = \pi_Y \circ t$.

Lemma A.2. *Consider a 2-commutative diagram of the form*

$$\begin{array}{ccccc} V & \longrightarrow & W & \longrightarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ U & \longrightarrow & X & \longrightarrow & Z \end{array}$$

and assume the right-hand small square to be 2-Cartesian. Then the left-hand small square is 2-Cartesian iff the big square is 2-Cartesian.

Proof. This is a standard fact, and we leave the proof to the reader. \square

We used the following easy consequence many times in Section 4.

Lemma A.3. Consider the following two diagrams of morphisms of stacks

$$\begin{array}{ccc} W & \xrightarrow{f} & Y \\ g \downarrow & & \downarrow h \\ X & \xrightarrow{j} & Z \end{array} \qquad \begin{array}{ccc} W & \xrightarrow{f} & Y \\ (g, k \circ f) \downarrow & & \downarrow (h, k) \\ X \times T & \xrightarrow{(j, 1)} & Z \times T \end{array}$$

Then if one is 2-Cartesian, so is the other.

Proof. This follows from Lemma A.2 once one knows that the square

$$\begin{array}{ccc} X \times T & \xrightarrow{(f, 1)} & Z \times T \\ \pi_X \downarrow & & \downarrow \pi_Z \\ X & \xrightarrow{f} & Z \end{array}$$

is 2-Cartesian. This follows from the diagram

$$\begin{array}{ccccc} X \times T & \xrightarrow{f \times 1} & Z \times T & \xrightarrow{\pi_T} & T \\ \pi_X \downarrow & & \pi_Z \downarrow & & \downarrow \\ X & \xrightarrow{f} & Z & \longrightarrow & \bullet \end{array}$$

by another application of Lemma A.2. \square

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