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Stability Conditions and Kleinian Singularities

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We describe (connected components of) the spaces of stability conditions on certain triangulated categories associated to Dynkin diagrams. These categories can be defined either algebraically via module categories of preprojective algebras, or geometrically via coherent sheaves on resolutions of Kleinian singularities. The resulting spaces of stability conditions are covering spaces of regular subsets of the corresponding complexified Cartan algebras.

1 Introduction

It is well known that many very different classes of mathematical objects are classified by Dynkin diagrams of ADE type. Thus, for example, to such a diagram Γ one can associate a finite-dimensional simple Lie algebra $\mathfrak{g}$, a braid group Br, a finite subgroup $G \subset SU_2$, a Kleinian surface singularity X, a complex reflection group W, a complex hyperplane arrangement, and so on. Of course, many connections between these objects are well known. For example, the surface singularity X is the quotient $\mathbb{C}^2/G$; the graph Γ can be recovered from the geometry of the minimal resolution of X and the McKay correspondence shows that one can equivalently recover Γ from the representation theory of G. Choosing a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ gives a root system $\Lambda \subset \mathfrak{h}^*$ whose associated reflection group is W; the complex hyperplane arrangement associated to Γ is then the

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complement of the complexified root hyperplanes $\mathfrak{h}^{\mathrm{reg}} \subset \mathfrak{h}$ and the braid group Br is the fundamental group of the space $\mathfrak{h}^{\mathrm{reg}}/W$.

In this paper, we show how some of the above objects relate to a certain triangulated category $\mathcal{D}$ associated to Γ. The precise definition of $\mathcal{D}$ is given in the next subsection; in fact, there are two equivalent definitions, one algebraic using the module category of the preprojective algebra of Γ and another geometric via coherent sheaves on the resolution of the Kleinian surface singularity X. We then show that a connected component of the space of stability conditions [2] on $\mathcal{D}$ is a covering space of $\mathfrak{h}^{\mathrm{reg}}$. We also prove that there is a natural action of the braid group Br on the category $\mathcal{D}$, which induces the group of deck transformations of this covering. Each of our results also has an analog in the affine case, where one replaces the graph Γ by the corresponding affine Dynkin diagram $\hat{\Gamma}$, the Lie algebra $\mathfrak{g}$ by the corresponding affine Kac–Moody Lie algebra $\hat{\mathfrak{g}}$, and so on.

We now proceed to a precise description of our results.

1.1 The categories

Let $G \subset \mathrm{SL}_2(\mathbb{C})$ be a finite subgroup, and let $\mathrm{Coh}_G(\mathbb{C}^2)$ denote the abelian category of G-equivariant coherent sheaves on $\mathbb{C}^2$. Let $\hat{\mathcal{A}} \subset \mathrm{Coh}_G(\mathbb{C}^2)$ denote the full subcategory consisting of equivariant sheaves supported at the origin in $\mathbb{C}^2$, and let $\mathcal{A} \subset \hat{\mathcal{A}}$ be the full subcategory consisting of equivariant sheaves with no nontrivial G-invariant sections. Let $\mathcal{D}$ and $\hat{\mathcal{D}}$ be the full subcategories of $\mathcal{D}^b\,\mathrm{Coh}_G(\mathbb{C}^2)$ consisting of complexes whose cohomology sheaves lie in $\mathcal{A}$ and $\hat{\mathcal{A}}$, respectively. The aim of this paper is to describe the spaces of stability conditions on these triangulated categories. Before describing the results in more detail, we give some alternative descriptions of our categories.

The first description is more geometric. Let $X = \mathbb{C}^2/G$ be the Kleinian quotient singularity associated to G, and let $f\colon Y \to X$ be the minimal resolution of singularities. The derived McKay correspondence [4, 11] gives an equivalence

$$\mathcal{D}^b\,\mathrm{Coh}_G(\mathbb{C}^2) \longrightarrow \mathcal{D}^b\,\mathrm{Coh}(Y).$$

It is easy to check that under this equivalence the subcategory $\hat{\mathcal{D}}$ corresponds to the full subcategory of $\mathcal{D}^b\,\mathrm{Coh}(Y)$ consisting of objects supported on the exceptional divisor $f^{-1}(0) \subset Y$, and the subcategory $\mathcal{D}$ corresponds to the full subcategory consisting of objects E satisfying $\mathbf{R}f_*(E) = 0$.

The second description is more algebraic. Recall that McKay [13] showed how to associate an extended Dynkin graph $\hat{\Gamma}$ to our finite subgroup $G \subset \mathrm{SL}_2(\mathbb{C})$. The vertices

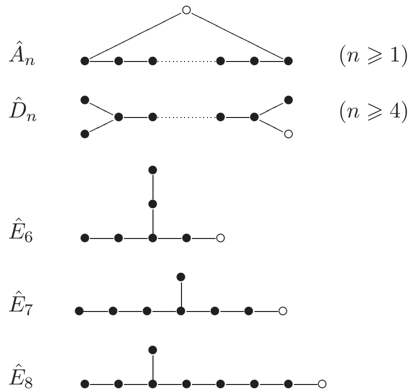

Fig. 1. The affine Dynkin diagrams $\hat{\Gamma}$.

of $\hat{\Gamma}$ are labelled by the isomorphism classes of irreducible representations of G, and the vertices corresponding to two irreducible representations ρ_i and ρ_j are joined by an edge precisely when $\rho_i \subset Q \otimes \rho_j$, where Q is the given representation $G \subset SL_2(\mathbb{C})$. The possible graphs $\hat{\Gamma}$ are shown in Fig. 1, with the special vertex corresponding to the trivial representation of G marked with an open dot. Removing this vertex leaves a Dynkin graph Γ.

The category $\mathrm{Coh}_G(\mathbb{C}^2)$ is tautologically the same thing as the category of modules for the skew group algebra $\mathbb{C}[x, y] * G$. In turn, it is known [7] that this algebra is Morita equivalent to the preprojective algebra $\hat{A}$ of the graph $\hat{\Gamma}$. More precisely, to define the preprojective algebra one must choose an orientation of $\hat{\Gamma}$, but different choices of orientation lead to isomorphic preprojective algebras. Using these identifications it is easy to see that $\hat{\mathcal{A}}$ is equivalent to the category of nilpotent modules for $\hat{A}$, and that $\mathcal{A}$ is equivalent to the full subcategory consisting of representations M satisfying $e_0 M = 0$, where $e_0 \in \hat{A}$ is the idempotent corresponding to the special vertex 0 of the quiver $\hat{\Gamma}$. From this description it is also immediate that $\mathcal{A}$ is equivalent to the category of finite-dimensional modules for the preprojective algebra A of the Dynkin quiver Γ.

The category $\hat{\mathcal{A}}$ is finite length and has $n+1$ simple objects $S_0, \ldots, S_n$ corresponding to the vertices of $\hat{\Gamma}$. In terms of equivariant coherent sheaves these simples are of the form $S_i = \rho_i \otimes \mathcal{O}_0$, where ρ_i is an irreducible representation of G and $\mathcal{O}_0$ is the skyscraper sheaf at the origin in $\mathbb{C}^2$. We shall always assume that S_0 corresponds to the trivial representation of G. The full subcategory $\mathcal{A} \subset \hat{\mathcal{A}}$ consists of those objects none of whose simple factors are isomorphic to S_0. Clearly, this is also a finite length category with simple objects $S_1, \ldots, S_n$ corresponding to the vertices of the graph Γ.

Remark 1.1. A standard argument shows that the category $\hat{\mathcal{D}}$ is equivalent to $\mathcal{D}^b(\hat{\mathcal{A}})$, although we shall not need this fact. However, the category $\mathcal{D}$ is definitely not equivalent to $\mathcal{D}^b(\mathcal{A})$. Indeed, the fact that $\hat{A}$ has finite global dimension implies that the category $\hat{\mathcal{D}}$ is of finite type, meaning that for any objects E and F

$$\dim_{\mathbb{C}}\left(\bigoplus_{i\in\mathbb{Z}} \operatorname{Hom}_{\hat{\mathcal{D}}}(E, F[i])\right) < \infty.$$

Since $\mathcal{D} \subset \hat{\mathcal{D}}$ is a subcategory, the same is true of $\mathcal{D}$. But the algebra A has infinite global dimension, so the category $\mathcal{D}^b(\mathcal{A})$ is not of finite type. In fact, the preprojective algebra of a graph has infinite global dimension precisely in the Dynkin case. (In the case when the graph has no loops, this follows immediately from [1, Proposition 4.2].) One could perhaps think of the category $\mathcal{D}$ as a homologically well-behaved substitute for the derived category of the preprojective algebra in the case when this algebra has infinite global dimension.

1.2 The finite-type case

Let $\mathfrak{g}$ be the finite-dimensional complex simple Lie algebra $\mathfrak{g}$ corresponding to the Dynkin graph Γ. Choose a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, and let $\Lambda \subset \mathfrak{h}^*$ be the corresponding finite root system. Let $\mathfrak{h}^{\text{reg}} \subset \mathfrak{h}$ be the complement of the root hyperplanes in $\mathfrak{h}$

$$\mathfrak{h}^{\text{reg}} = \{\theta \in \mathfrak{h} : \theta(\alpha) \neq 0 \text{ for all } \alpha \in \Lambda\}.$$

The Weyl group W is generated by reflections in the root hyperplanes and acts freely on $\mathfrak{h}^{\text{reg}}$.

The Grothendieck group $K(\mathcal{D})$ equipped with the Euler form can be identified with the root lattice $\mathbb{Z}\Lambda$ equipped with the Killing form. The simple objects $S_i \in \mathcal{A}$ are spherical objects in $\mathcal{D}$ and hence by results of Seidel and Thomas [15] define autoequivalences $\Phi_{S_i} \in \operatorname{Aut}(\mathcal{D})$. We write $\operatorname{Br}(\mathcal{D})$ for the subgroup of $\operatorname{Aut}(\mathcal{D})$ they generate.

The following result generalises a result of Thomas [16] who proved the A_n case using different methods. In fact, Thomas worked with a triangulated category whose objects were dg-modules over a dg-algebra, but the formality result of [15] shows that his category is equivalent to ours (see [16, Section 3]).

Theorem 1.2. There is a connected component $\operatorname{Stab}^\dagger(\mathcal{D}) \subset \operatorname{Stab}(\mathcal{D})$ that is a covering space of $\mathfrak{h}^{\mathrm{reg}}/W$. The subgroup $\operatorname{Br}(\mathcal{D}) \subset \operatorname{Aut}(\mathcal{D})$ preserves this component and acts as the group of deck transformations.

The fundamental group of the quotient $\mathfrak{h}^{\mathrm{reg}}/W$ coincides [5, 8] with the braid group $\operatorname{Br}(\Gamma)$ of the graph Γ. This is the group generated by elements $\sigma_1, \dots, \sigma_n$ indexed by the vertices subject to relations $\sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j$ if the vertices i and j are connected by an edge, and $\sigma_i\sigma_j = \sigma_j\sigma_i$ otherwise. It follows from Theorem 1.2 that there is a surjective homomorphism

$$\rho\colon \operatorname{Br}(\Gamma) \longrightarrow \operatorname{Br}(\mathcal{D}).$$

With further thought one can show that ρ can be taken to send the generator σ_i to the twist functor Φ_{S_i}. In the A_n case, Seidel and Thomas [15] were able to show that ρ is an isomorphism, so that $\operatorname{Stab}^\dagger(\mathcal{D})$ is actually the universal cover of $\mathfrak{h}^{\mathrm{reg}}/W$. Reversing this argument, one might hope to find a general proof that $\operatorname{Stab}^\dagger(\mathcal{D})$ is simply-connected; this would then imply that ρ is always an isomorphism.

Theorem 1.2 allows us to say something about the group of autoequivalences of the category $\mathcal{D}$. Unfortunately, we cannot rule out the possibility of exotic autoequivalences, which permute the connected components of the space of stability conditions. So define $\operatorname{Aut}^\dagger(\mathcal{D})$ to be the subgroup of autoequivalences which preserve the connected component $\operatorname{Stab}^\dagger(\mathcal{D})$ of Theorem 1.1. A further problem is that $\operatorname{Aut}^\dagger(\mathcal{D})$ could in theory contain autoequivalences Φ which act trivially, in the sense that $\Phi(E) \cong E$ for all objects E. In fact, we are really interested only in the action of $\operatorname{Aut}^\dagger(\mathcal{D})$ on $\operatorname{Stab}^\dagger(\mathcal{D})$. So define $\operatorname{Aut}^\dagger_*(\mathcal{D})$ to be the quotient $\operatorname{Aut}^\dagger(\mathcal{D})/H$, where H is the subgroup of $\operatorname{Aut}^\dagger(\mathcal{D})$ consisting of autoequivalences, which fix every point of $\operatorname{Stab}^\dagger(\mathcal{D})$.

Corollary 1.3. Let $\operatorname{Aut}(\Gamma)$ be the group of symmetries of the graph Γ. There is an isomorphism

$$\operatorname{Aut}^\dagger_*(\mathcal{D}) \cong \operatorname{Br}(\mathcal{D}) \rtimes \operatorname{Aut}(\Gamma),$$

where $\operatorname{Aut}(\Gamma)$ acts on $\operatorname{Br}(\mathcal{D})$ by permuting the generators Φ_{S_i}.

One might wonder where the shift functor has gone in Corollary 1.3, but one can easily check by direct computation that, for example, in the A_n case, one has

$$\left(\Phi_{S_1} \circ \Phi_{S_2} \circ \cdots \circ \Phi_{S_n}\right)^{n+1} = [-2],$$

with the obvious numbering of the vertices of Γ.

1.3 The affine case

Let $\hat{\mathfrak{g}}$ be the affine Kac–Moody Lie algebra associated to the graph $\hat{\Gamma}$. The roots $\hat{\Lambda}$ of $\hat{\mathfrak{g}}$ span an $(n+1)$-dimensional subspace $\hat{\mathfrak{h}}^* \subset \hat{\mathfrak{g}}^*$. Restricting the nondegenerate symmetric form on $\hat{\mathfrak{g}}^*$ obtained from the invariant form on $\hat{\mathfrak{g}}$ gives a degenerate symmetric form on $\hat{\mathfrak{h}}^*$. The Grothendieck group $K(\hat{\mathcal{D}})$ equipped with the Euler form can then be identified with the root lattice $\mathbb{Z}\hat{\Lambda}$ equipped with this form. Let $\hat{\mathfrak{h}}$ be the dual space to $\hat{\mathfrak{h}}^*$, and let $\hat{\mathfrak{h}}^{\text{reg}} \subset \hat{\mathfrak{h}}$ be the complement of the affine root hyperplanes in $\hat{\mathfrak{h}}$

$$\hat{\mathfrak{h}}^{\text{reg}} = \{\theta \in \hat{\mathfrak{h}} : \theta(\alpha) \neq 0 \text{ for all } \alpha \in \hat{\Lambda}\}.$$

It is easy to see that this is an open subset of $\hat{\mathfrak{h}}$. The affine Weyl group $\hat{W}$ is generated by reflections in the real root hyperplanes and acts freely on $\hat{\mathfrak{h}}^{\text{reg}}$.

Once again, the simple objects $S_i \in \hat{\mathcal{A}}$ are spherical objects in $\hat{\mathcal{D}}$ and hence define autoequivalences $\Phi_{S_i} \in \operatorname{Aut}(\hat{\mathcal{D}})$. We write $\operatorname{Br}(\hat{\mathcal{D}})$ for the subgroup of $\operatorname{Aut}(\hat{\mathcal{D}})$ they generate. This time, $\operatorname{Br}(\hat{\mathcal{D}})$ does not contain any power of the shift functor so we also consider the subgroup $\operatorname{Br}(\hat{\mathcal{D}}) \times \mathbb{Z} \subset \operatorname{Aut}(\hat{\mathcal{D}})$, where the second factor is generated by the shift functor [2].

Theorem 1.4. There is a connected component $\operatorname{Stab}^{\dagger}(\hat{\mathcal{D}}) \subset \operatorname{Stab}(\hat{\mathcal{D}})$ that is a covering space of $\hat{\mathfrak{h}}^{\text{reg}}/\hat{W}$. The subgroup $\operatorname{Br}(\hat{\mathcal{D}}) \times \mathbb{Z} \subset \operatorname{Aut}(\hat{\mathcal{D}})$ preserves this component and acts as the group of deck transformations.

There is a $\hat{W}$ invariant map $\hat{\mathfrak{h}}^{\text{reg}} \to \mathbb{C}^*$ obtained by evaluating an element θ on an imaginary root δ. The fundamental group of the fibre of this map coincides [14] with the braid group $\operatorname{Br}(\hat{\Gamma})$ associated to the graph $\hat{\Gamma}$. There is thus a short exact sequence

$$1 \longrightarrow \operatorname{Br}(\hat{\Gamma}) \longrightarrow \pi_1(\hat{\mathfrak{h}}^{\text{reg}}/\hat{W}) \longrightarrow \mathbb{Z} \longrightarrow 1,$$

which can be split by taking a loop γ of the form $\theta(t) = \theta_0 \cdot e^{2\pi i t}$. It follows from Theorem 1.4 that there is a surjective homomorphism

$$\pi_1(\hat{\mathfrak{h}}^{\text{reg}}/\hat{W}) \longrightarrow \operatorname{Br}(\hat{\mathcal{D}}) \times \mathbb{Z},$$

and the proof shows that this induces a surjective homomorphism

$$\rho\colon \operatorname{Br}(\hat{\Gamma}) \longrightarrow \operatorname{Br}(\hat{\mathcal{D}}).$$

As before, define $\operatorname{Aut}^{\dagger}(\hat{\mathcal{D}})$ to be the subgroup of autoequivalences which preserve the connected component $\operatorname{Stab}^{\dagger}(\hat{\mathcal{D}})$ of Theorem 1.4, and $\operatorname{Aut}_*^{\dagger}(\hat{\mathcal{D}})$ to be the quotient by the autoequivalences which act trivially on $\operatorname{Stab}^{\dagger}(\hat{\mathcal{D}})$. Then we have the following corollary.

Corollary 1.5. Let $\operatorname{Aut}(\hat{\Gamma})$ be the group of symmetries of the graph $\hat{\Gamma}$. There is an isomorphism

$$\operatorname{Aut}_*^{\dagger}(\hat{\mathcal{D}}) = \mathbb{Z} \times (\operatorname{Br}(\hat{\mathcal{D}}) \rtimes \operatorname{Aut}(\hat{\Gamma})),$$

where the factor of $\mathbb{Z}$ is generated by the shift [1], and $\operatorname{Aut}(\hat{\Gamma})$ acts on $\operatorname{Br}(\hat{\mathcal{D}})$ by permuting the generators Φ_{S_i}.

A more careful analysis of the group $\operatorname{Aut}(\hat{\mathcal{D}})$ has been carried out in the $\hat{A}_n$ case by Ishii and Uehara [9].

2 Background

2.1 The Categories

Here we state some simple facts about the categories defined in the Introduction, which will be used in the proofs of our main theorems. These results are all well known and we only sketch the proofs. Further details can be found in [7].

Lemma 2.1. The categories $\mathcal{D}$ and $\hat{\mathcal{D}}$ are triangulated categories of finite type with Serre functor [2].

Proof. As explained in the Introduction, we can identify $\hat{\mathcal{D}}$ with the full subcategory of the derived category of coherent sheaves on the minimal resolution $f\colon Y \to \mathbb{C}^2/G$ consisting of objects supported on the exceptional fibre $f^{-1}(0)$. It follows that $\hat{\mathcal{D}}$ has finite type. Since $Y \to \mathbb{C}^2/G$ is crepant, the canonical bundle ω_Y is trivial on this fibre. The result then follows from Serre duality. ∎

Recall that an object $S \in \hat{\mathcal{D}}$ is spherical if

$$\operatorname{Hom}^k_{\hat{\mathcal{D}}}(S, S) = \begin{cases} \mathbb{C} & \text{if } k = 0 \text{ or } 2, \\ 0 & \text{otherwise.} \end{cases}.$$

It follows from constructions given in [15] that any such object defines an autoequivalence $\Phi_S \in \operatorname{Aut} \hat{\mathcal{D}}$ called a twist functor such that for any $E \in \hat{\mathcal{D}}$ there is a triangle

$$\operatorname{Hom}_{\hat{\mathcal{D}}}(S, E) \otimes S \longrightarrow E \longrightarrow \Phi_S(E).$$

Note that at the level of the Grothendieck group the functor induces a reflection

$$\phi_S([E]) = [E] - \chi([S], [E])[S].$$

These twist functors will be very important in what follows.

Lemma 2.2. The abelian category $\mathcal{A}$ is of finite length with simple objects $S_1, \ldots, S_n$ labelled by the vertices of the graph Γ. Each of these objects is spherical in $\mathcal{D}$. Given any two of these simples, the space $\operatorname{Hom}^1_{\mathcal{D}}(S_i, S_j)$ is one or zero-dimensional depending on whether the corresponding vertices of Γ are joined by an edge or not.

Proof. This is an easy computation of Ext-groups, either in the category $\operatorname{Coh}_G(\mathbb{C}^2)$ or in the category of representations of the preprojective algebra. ■

It follows from Lemma 2.2 that $\chi(S_i, S_i) = 2$ for all i and $\chi(S_i, S_j) = 0$ or 1 depending on whether i and j are connected by an edge in Γ. Thus, the Grothendieck group $K(\mathcal{D})$ with its Euler form can be identified with the root lattice $\mathbb{Z}\Lambda \subset \mathfrak{h}^*$ of the corresponding finite-dimensional Lie algebra $\mathfrak{g}$, equipped with the unique multiple of the Killing form for which $(\alpha, \alpha) = 2$ for all roots $\alpha \in \Lambda$. Under this identification, the classes of spherical objects of $\mathcal{D}$ correspond to the roots Λ and the classes $[S_i]$ form a system of simple roots. The reflection ϕ_S of $K(\mathcal{D})$ defined by a spherical object S of $\mathcal{D}$ induces the root reflection of the root lattice defined by the corresponding element of Λ.

Note that in the terminology of [10] the finite-dimensional Lie algebra $\mathfrak{g}$ is the Kac–Moody Lie algebra with generalised Cartan matrix $a_{ij} = \chi(S_i, S_j)$ for $1 \leqslant i, j \leqslant n$.

Lemma 2.3. The abelian category $\hat{\mathcal{A}}$ is of finite length with simple objects $S_0, \ldots, S_n$ labelled by the vertices of the graph $\hat{\Gamma}$. Each of these objects is spherical in $\hat{\mathcal{D}}$. Given any two of these simples, the dimension of the space $\operatorname{Hom}^1_{\hat{\mathcal{D}}}(S_i, S_j)$ is the number of edges connecting the corresponding vertices of the graph $\hat{\Gamma}$.

Proof. This is the same as the proof of Lemma 2.2. ■

Note that the number of edges connecting two vertices in the graph $\hat{\Gamma}$ must be either zero or one, with the single exception of the graph $\hat{A}_1$ which consists of two vertices joined by two edges.

As before, Lemma 2.3 is enough to calculate the Euler form on the Grothendieck group $K(\hat{\mathcal{D}})$. This time the form is indefinite with a one-dimensional kernel generated by the class of the equivariant sheaf $\mathbb{C}[G] \otimes \mathcal{O}_0$. The combinatorics of the category $\hat{\mathcal{D}}$ are described by the root system of the affine Kac–Moody Lie algebra $\hat{\mathfrak{g}}$ corresponding to the graph $\hat{\Gamma}$, although one has to be a little careful here because this Lie algebra has a degenerate Cartan matrix.

Thus, let $\hat{\mathfrak{g}}$ be the affine Kac–Moody Lie algebra with generalised Cartan matrix $a_{ij} = \chi(S_i, S_j)$ (see [10] for the definition). There is a decomposition

$$\hat{\mathfrak{g}} = \hat{\mathfrak{g}}_0 \oplus \bigoplus_{\alpha \in \hat{\Lambda}} \hat{\mathfrak{g}}_\alpha,$$

where $\hat{\Lambda} \subset \hat{\mathfrak{g}}_0^*$ is the (infinite) set of affine roots. The abelian subalgebra $\hat{\mathfrak{g}}_0$ has dimension $n+2$. The roots $\hat{\Lambda}$ span a codimension one subspace which we will denote $\hat{\mathfrak{h}}^* \subset \hat{\mathfrak{g}}_0^*$. Note that this differs from the notation of [10] where $\hat{\mathfrak{h}}^* = \hat{\mathfrak{g}}_0^*$. The nondegenerate invariant inner product on $\hat{\mathfrak{g}}$ restricts to a nondegenerate inner product on $\hat{\mathfrak{g}}_0$ and hence induces a nondegenerate inner product $(-,-)$ on $\hat{\mathfrak{g}}_0^*$. On roots this form is given by $(\alpha_i, \alpha_j) = a_{ij}$ (see [10, (2.1.6)]). Thus, we can identify the root lattice $\mathbb{Z}\hat{\Lambda}$ equipped with the inner product $(-,-)$ with the Grothendieck group $K(\hat{\mathcal{D}})$ equipped with the Euler form.

Under this identification, the classes of the simple objects $S_0, \cdots, S_n$ correspond to a set of simple roots. The classes of the spherical objects of $\hat{\mathcal{D}}$ correspond to the real roots in Λ, and the class of the equivariant sheaf $\mathbb{C}[G] \otimes \mathcal{O}_0$ corresponds to an imaginary root δ. Once again, the reflection ϕ_S of $K(\hat{\mathcal{D}})$ defined by a spherical object S of $\hat{\mathcal{D}}$ induces the reflection of the root lattice defined by the corresponding real root.

2.2 Stability Conditions

We refer the reader to [2] for definitions concerning stability conditions. Here we just give a brief summary, mainly in order to fix notation.

A stability condition $\sigma = (Z, \mathcal{P})$ on a triangulated category $\mathcal{D}$ is defined by full abelian subcategories $\mathcal{P}(\phi) \subset \mathcal{D}$ for each $\phi \in \mathbb{R}$ together with a group homomorphism $Z\colon K(\mathcal{D}) \to \mathbb{C}$ having the property that

$$0 \neq E \in \mathcal{P}(\phi) \implies Z(E) \in \mathbb{R}_{>0} \exp(i\pi\phi).$$

The map Z is called the central charge, the nonzero objects of $\mathcal{P}(\phi)$ are called the semistables of phase ϕ, and the simple objects are stable. The smallest extension-closed subcategory of $\mathcal{D}$ containing the objects of $\mathcal{P}(\phi)$ for each $\phi \in (0, 1]$ is an abelian category called the heart of the stability condition σ.

In fact, σ is completely determined by its heart together with the central charge Z, and conversely, if $\mathcal{A} \subset \mathcal{D}$ is the heart of a bounded t-structure, and $Z\colon K(\mathcal{D}) \to \mathbb{C}$ is a group homomorphism with the property that

$$0 \neq E \in \mathcal{A} \implies Z(E) \in \mathbb{R}_{>0} \exp(i\pi\phi) \text{ with } \phi \in (0, 1], \tag{$*$}$$

then there is a stability condition σ on $\mathcal{D}$ with heart $\mathcal{A}$ and central charge Z, providing that $\mathcal{A}$ satisfies a certain Harder–Narasimhan property with respect to Z. This property is automatically satisfied if $\mathcal{A}$ has finite length.

Let us assume that $K(\mathcal{D})$ is of finite rank. The set of all stability conditions satisfying a technical condition called local-finiteness then form a complex manifold $\operatorname{Stab}(\mathcal{D})$. There is a continuous forgetful map

$$\mathcal{Z}\colon \operatorname{Stab}(\mathcal{D}) \longrightarrow \operatorname{Hom}_{\mathbb{Z}}(K(\mathcal{D}), \mathbb{C})$$

sending a stability condition to its central charge.

The group of exact autoequivalences $\operatorname{Aut}(\mathcal{D})$ of $\mathcal{D}$ act on $\operatorname{Stab}(\mathcal{D})$: an element $\Phi \in \operatorname{Aut}(\mathcal{D})$ sends $(Z, \mathcal{P})$ to $(Z', \mathcal{P}')$, where $\mathcal{P}'(\phi) = \Phi(\mathcal{P}(\phi))$ and $Z'(E) = Z(\Phi^{-1}(E))$. The additive group $\mathbb{C}$ also acts on $\operatorname{Stab}(\mathcal{D})$: an element $\lambda \in \mathbb{C}$ sends $(Z, \mathcal{P})$ to $(Z', \mathcal{P}')$, where $\mathcal{P}'(\phi) = \mathcal{P}(\phi + \operatorname{Re}(\lambda))$ and $Z'(E) = \exp(-i\pi\lambda)Z(E)$. These two actions commute, and the action of the shift functor [1] coincides with the action of $1 \in \mathbb{C}$.

3 The Results

In this section, we prove Theorem 1.4 and Corollary 1.5 as stated in the Introduction. The corresponding results of Theorem 1.2 and Corollary 1.3 for the Dynkin case are entirely analogous, but easier, and we shall confine ourselves to a few remarks on the proof at the end.

Recall that we can identify the Grothendieck group $K(\hat{\mathcal{D}})$ with the affine root lattice $\mathbb{Z}\hat{\Lambda} \subset \hat{\mathfrak{h}}^*$. The classes $\alpha_i = [S_i]$ of the $n+1$ simple objects $S_i \in \hat{\mathcal{A}}$ define a set of simple roots in $\hat{\Lambda}$. The class of the equivariant sheaf $\mathbb{C}[G] \otimes \mathcal{O}_0$ corresponds to an imaginary root δ. We can identify group homomorphisms $Z\colon K(\hat{\mathcal{D}}) \to \mathbb{C}$ with elements of $\hat{\mathfrak{h}}$. Thus, we have a continuous map $\mathcal{Z}\colon \operatorname{Stab}(\hat{\mathcal{D}}) \to \hat{\mathfrak{h}}$ sending a stability condition to its central charge.

Lemma 3.1. For each point Z in the complexified Weyl chamber,

$$R = \{Z \in \hat{\mathfrak{h}} : \operatorname{Im} Z(\alpha_i) > 0 \text{ for all } i\} \subset \hat{\mathfrak{h}}$$

there is a unique stability condition $\sigma \in \operatorname{Stab}(\hat{\mathcal{D}})$ with heart $\hat{\mathcal{A}}$ and central charge Z. These points form a region $U \subset \operatorname{Stab}(\hat{\mathcal{D}})$, which is mapped homeomorphically by $\mathcal{Z}$ onto R.

Proof. The standard t-structure on $\mathcal{D}^b \operatorname{Coh}_G(\mathbb{C}^2)$ induces a bounded t-structure on $\hat{\mathcal{D}}$ with heart $\hat{\mathcal{A}}$. Since $\hat{\mathcal{A}}$ has finite length, the class of any nonzero element $E \in \hat{\mathcal{A}}$ is a positive linear combination of the simple roots α_i. Thus, the condition $(*)$ holds. The Harder–Narasimhan property is automatic because $\hat{\mathcal{A}}$ has finite length. The resulting stability condition $\sigma = (Z, \mathcal{P})$ is locally finite because for any ϕ there is an $\epsilon > 0$ such that the subcategories $\mathcal{P}((\phi - \epsilon, \phi + \epsilon))$ are contained in some shift of $\hat{\mathcal{A}}$ and hence are of finite length. ∎

We shall need a result which follows from work of Craw and Ishii on moduli of G-constellations [6, Proposition 2.2].

Lemma 3.2. For each point $\sigma \in U$, there is a semistable object whose class in $K(\hat{\mathcal{D}}) = \mathbb{Z}\hat{\Lambda}$ is the imaginary root δ.

Proof. A G-constellation is a representation of the group ring $\mathbb{C}[x, y] * G$ which as a $\mathbb{C}[G]$-module is isomorphic to $\mathbb{C}[G]$. Craw and Ishii use general results of King [12] to show that for any generic choice of weights the moduli space of semistable G-constellations is nonempty and has a projective morphism to the quotient $X = \mathbb{C}^2/G$. The fibre over the origin is then nonempty and consists of nilpotent representations. These define objects of $\hat{\mathcal{A}}$ whose class in $K(\hat{\mathcal{D}})$ is δ. ∎

Now we need two general results from [3]. Let $\operatorname{Stab}^{\dagger}(\hat{\mathcal{D}})$ be the connected component of $\operatorname{Stab}(\hat{\mathcal{D}})$ containing the subset U.

Proposition 3.3. The map $\mathcal{Z} : \operatorname{Stab}^{\dagger}(\hat{\mathcal{D}}) \to \hat{\mathfrak{h}}$ sending a stability condition to its central charge is a local homeomorphism onto an open subset of $\hat{\mathfrak{h}}$ containing $\hat{\mathfrak{h}}^{\text{reg}}$. The restriction to $\mathcal{Z}^{-1}(\hat{\mathfrak{h}}^{\text{reg}})$ is a covering map.

Proof. It is a general result [2] that $\mathcal{Z}$ is a local homeomorphism onto an open subset of some linear subspace of $\hat{\mathfrak{h}}$. Since this subspace contains R, it must be the whole of $\hat{\mathfrak{h}}$. The fact that the restriction is a covering map is proved in exactly the same way as the corresponding result on coherent sheaves on K3 surfaces [3, Section 8]. This then implies that the image contains $\hat{\mathfrak{h}}^{\mathrm{reg}}$. ■

Proposition 3.4. Let $E \in \hat{\mathcal{D}}$ be stable in a stability condition $\sigma \in \operatorname{Stab}^{\dagger}(\hat{\mathcal{D}})$. Then there is an open neighborhood $\sigma \in N \subset \operatorname{Stab}^{\dagger}(\hat{\mathcal{D}})$ such that E is stable for all stability conditions in N.

Proof. Again this is exactly the same argument as in the K3 surface case (see [3, Section 9]). In fact the only property of $\operatorname{Stab}^{\dagger}(\hat{\mathcal{D}})$ needed is that it contains points σ satisfying $\mathcal{Z}(\sigma) \in \hat{\mathfrak{h}}^{\mathrm{reg}}$. ■

The following result shows that the autoequivalences Φ_{S_i} preserve the connected component $\operatorname{Stab}^{\dagger}(\hat{\mathcal{D}})$ so that the group $\operatorname{Br}(\hat{\mathcal{D}})$ acts on $\operatorname{Stab}^{\dagger}(\hat{\mathcal{D}})$.

Lemma 3.5. Let $\sigma = (Z, \mathcal{P})$ be a point in the boundary of U for which there is a unique simple $S_i \in \hat{\mathcal{A}}$ with $\operatorname{Im} Z(S_i) = 0$. Assume further that $Z(S_i) \in \mathbb{R}_{<0}$. Then the stability condition $\Phi_{S_i}^{-1}(\sigma)$ also lies in the boundary of U.

Proof. To help with the notation set $T = S_i$. Take a small neighborhood V of σ in $\operatorname{Stab}(\hat{\mathcal{D}})$ and consider the open subset

$$V_+ = \{\tau = (Z, \mathcal{P}) \in V : \operatorname{Im} Z(T) < 0\}.$$

We claim that we can choose V small enough so that $\Phi_T^{-1}(V_+) \subset U$. It follows that the stability condition $\Phi_T^{-1}(\sigma)$ lies in the closure of U. It cannot lie in U because $Z(\Phi_T(T)) = Z(T[-1])$ lies on the positive real axis.

Thus, we are required to prove that if V is small enough, the heart of any $\tau \in V_+$ is equal to $\Phi_T(\hat{\mathcal{A}}) \subset \hat{\mathcal{D}}$. It is a simple fact that if $\mathcal{C}$ and $\mathcal{C}'$ are both hearts of bounded t-structures in a triangulated category and $\mathcal{C} \subset \mathcal{C}'$, then $\mathcal{C} = \mathcal{C}'$. Since $\hat{\mathcal{A}}$ has finite length, it will therefore be enough to prove that for all j the object $\Phi_T(S_j)$ lies in the heart of any $\tau \in V_+$.

Assume first that $j \neq i$. If vertices j and i are joined by an edge in $\hat{\Gamma}$, then $\operatorname{Hom}^1_{\hat{\mathcal{D}}}(S_i, S_j) = \mathbb{C}$ so there is a nonsplit short exact sequence in $\hat{\mathcal{A}}$

$$0 \longrightarrow S_j \longrightarrow \Phi_T(S_j) \longrightarrow T \longrightarrow 0.$$

It follows that $\Phi_T(S_j)$ is in the heart of σ and its semistable factors have phases in the interval $(0, 1)$. Choosing V small enough, we can assume that this is the case for all $\tau \in V$ too. If i and j are not joined by an edge, then $\Phi_T(S_j) = S_j$ and the same argument applies.

Finally, consider $\Phi_T(T) = T[-1]$. Since T was stable in σ with phase 1, we can assume that T is also stable for all $\tau \in V$, with phase 2 at most. Clearly, one must have $\phi(T) > 1$ for $\tau \in V_+$. This implies that $T[-1]$ has phase in the interval $(0, 1]$ and hence lies in the heart of τ. ∎

Lemma 3.5 shows that the autoequivalence Φ_{S_i} exchanges the two pieces of the boundary of U given by $Z(S_i) \in \mathbb{R}_{<0}$ and $Z(S_i) \in \mathbb{R}_{>0}$. The crucial thing is that this autoequivalence reverses the orientations, taking the side where $\operatorname{Im} Z(S_i) > 0$ to the side where $\operatorname{Im} Z(S_i) < 0$. This observation easily gives the following lemma.

Lemma 3.6. For every stability condition $\sigma = (Z, \mathcal{P}) \in \operatorname{Stab}^\dagger(\hat{\mathcal{D}})$, the central charge Z does not vanish on the imaginary root δ. Furthermore, there is an autoequivalence $\Phi \in \operatorname{Br}(\hat{\mathcal{D}})$ and an element $\lambda \in \mathbb{C}$ such that $\lambda\Phi(\sigma)$ lies in the closure of U.

Proof. First assume that the central charge Z of σ does not vanish on the imaginary roots δ. Choose a path γ joining σ to a point of U. Since $\mathcal{Z}$ is a local homeomorphism, we can assume that $Z(\delta) \neq 0$ for all stability conditions on the path γ. Normalizing with the $\mathbb{C}$ action on $\operatorname{Stab}(\hat{\mathcal{D}})$, we can replace σ by some $\lambda(\sigma)$ and assume that γ lies in the affine slice

$$\hat{\mathfrak{h}}_a^{\text{reg}} = \{\theta \in \hat{\mathfrak{h}}^{\text{reg}} : \theta(\delta) = i\}.$$

In this slice, the complexified Weyl alcoves are locally finite, and since $\mathcal{Z}$ is a local homeomorphism, we can wiggle the path γ a bit so that it passes through finitely many Weyl alcoves, and only passes through codimension one walls. Each time γ passes through a wall, Lemma 3.5 shows that there is an element of $\operatorname{Br}(\hat{\mathcal{D}})$ that takes one back to a stability condition in the closure of U. The result then follows.

Now suppose $Z(\delta) = 0$. In particular, there are no semistable objects in σ whose class in $K(\hat{\mathcal{D}})$ is δ. By the results of [3, Section 9], this is true in an open neighborhood of σ in $\operatorname{Stab}^\dagger(\hat{\mathcal{D}})$. But by the first part and Lemma 3.2, stability conditions near σ for which $Z(\delta) \neq 0$ do have semistable objects with class δ. This gives a contradiction. ∎

Now we can prove our main results.

Proof of Theorem 1.4. First we show that the image of the map $\mathcal{Z}$ is contained in $\hat{\mathfrak{h}}^{\text{reg}}$. Proposition 3.3 then shows that $\mathcal{Z}$ is a covering map. Note that the autoequivalences Φ_{S_i} act on $K(\mathcal{D})$ as root reflections in the simple roots α_i. Thus, the action of $\text{Br}(\hat{\mathcal{D}})$ on $\text{Stab}^{\dagger}(\hat{\mathcal{D}})$ induces the action of the affine Weyl group on $\hat{\mathfrak{h}}$, which preserves $\hat{\mathfrak{h}}^{\text{reg}}$. Similarly, the action of $\mathbb{C}$ descends to the rescaling action of $\mathbb{C}^*$, which also preserves $\hat{\mathfrak{h}}^{\text{reg}}$.

Take a point $\sigma = (Z, \mathcal{P}) \in \text{Stab}^{\dagger}(\hat{\mathcal{D}})$. By the last result, we can assume that Z lies in the affine slice $\hat{\mathfrak{h}}_a^{\text{reg}}$. This slice has a locally-finite decomposition into closures of complexified Weyl alcoves of the form

$$\{\theta \in \hat{\mathfrak{h}} : \text{Im}\, \theta(\alpha_i) \geqslant 0 \text{ for all } i\} \subset \hat{\mathfrak{h}},$$

corresponding to a choice of simple roots $\alpha_1, \ldots, \alpha_n \in \hat{\Lambda}$. Suppose $Z \notin \hat{\mathfrak{h}}^{\text{reg}}$ so that $Z(\alpha) = 0$ for some real root α. We claim that Z lies in the boundary of an alcove corresponding to a set of simple roots containing α. Indeed, otherwise none of the boundaries of Weyl alcoves passing through σ are given by $\text{Im}\, Z(\alpha) = 0$, so we can deform Z into an alcove preserving the condition $Z(\alpha) = 0$. This is impossible since each alcove is contained in $\hat{\mathfrak{h}}^{\text{reg}}$. Thus, σ lies in the boundary of a subset of the form $U \subset \text{Stab}(\hat{\mathcal{D}})$ corresponding to a heart $\hat{\mathcal{A}} \subset \hat{\mathcal{D}}$ containing a simple object of class α. This object S is stable for all stability conditions in U and hence semistable in σ which contradicts the assumption that $Z(\alpha) = 0$. Thus, we conclude that the image of $\mathcal{Z}$ is contained in $\hat{\mathfrak{h}}^{\text{reg}}$.

By what was said above, the group $\text{Br}(\hat{\mathcal{D}})$ acts as deck transformations for the covering map $\text{Stab}^{\dagger}(\hat{\mathcal{D}}) \to \hat{\mathfrak{h}}^{\text{reg}}/\hat{W}$. Conversely, suppose two points $\sigma_1, \sigma_2 \in \text{Stab}^{\dagger}(\hat{\mathcal{D}})$ map to the same point in $\hat{\mathfrak{h}}^{\text{reg}}/\hat{W}$ and assume $\sigma_1 \in U$. Applying Lemma 3.6 shows that there is an element $\Phi \in \text{Br}(\hat{\mathcal{D}})$ and a $\lambda \in \mathbb{C}$ such that $\lambda\Phi(\sigma_2) \in \bar{U}$. But since the complexified Weyl chamber R is a fundamental domain for the action of $\hat{W}$ on $\hat{\mathfrak{h}}$, which commutes with the $\mathbb{C}^*$-action, it follows that $\lambda = 2n$ is an even integer and $\sigma_1 = \Phi(\sigma_2)[2n]$. ■

Proof of Corollary 1.5. Suppose an autoequivalence Ψ of $\hat{\mathcal{D}}$ preserves the connected component $\text{Stab}^{\dagger}(\hat{\mathcal{D}})$. Take a stability condition $\sigma \in U$ and consider the stability condition $\Psi(\sigma)$. By Lemma 3.6, there is an element $\Phi \in \text{Br}(\hat{\mathcal{D}})$ and a $\tau \in \bar{U}$ such that $\tau = \lambda\Phi\Psi(\sigma)$ for some $\lambda \in \mathbb{C}$. Suppose we chose $\sigma \in U$ so that $\text{Re}\, Z(S_i) = 0$ for all i. Then there is some shift $[n]$ such that the stability condition $\sigma' = \lambda(\sigma)[-n]$ lies in U. Now $\Upsilon = \Phi\Psi[n]$ takes σ' to τ. Deforming σ' and τ a bit, we can assume that they both lie in U and hence have heart $\hat{\mathcal{A}}$. It follows that Υ fixes $\hat{\mathcal{A}} \subset \hat{\mathcal{D}}$.

Now Υ permutes the simple objects of $\hat{\mathcal{A}}$ and hence induces an automorphism of the graph $\hat{\Gamma}$. Viewing $\hat{\mathcal{D}}$ as a subcategory of the derived category of representations of the preprojective algebra of $\hat{\Gamma}$, it is easy to see that conversely any automorphism of $\hat{\Gamma}$ lifts to an automorphism of $\hat{\mathcal{D}}$ preserving $\hat{\mathcal{A}}$. Thus, we may assume that Υ fixes the simples S_i. But then it acts trivially on $K(\hat{\mathcal{D}})$ and hence fixes all stability conditions $\sigma \in U$. It follows that it acts trivially on $\operatorname{Stab}_0(\hat{\mathcal{D}})$. ∎

The proofs in the finite type cases proceed in exactly the same way. Each result we proved in this section holds also for the Dynkin case on doffing hats and replacing the $n+1$ simples $S_0, \ldots, S_n$ of $\hat{\mathcal{A}}$ with the n simples $S_1, \ldots, S_n$ of $\mathcal{A}$. The only exception is Lemma 3.2 which is meaningless in the Dynkin case. The proof of Lemma 3.6 is easier since the number of Weyl chambers is finite so we have no need to rescale or pass to an affine slice.

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