

STABILITY CONDITIONS ON K3 SURFACES

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ABSTRACT. This paper contains a description of one connected component of the space of stability conditions on the bounded derived category of coherent sheaves on a complex algebraic K3 surface.

1. INTRODUCTION

The notion of a stability condition on a triangulated category was introduced in [5] in an effort to understand M. Douglas' work on π -stability for D-branes [7]. It was shown in [5] that to any triangulated category \mathcal{D} , one can associate a complex manifold $\text{Stab}(\mathcal{D})$ parameterising stability conditions on \mathcal{D} . Douglas' work suggests that spaces of stability conditions should be closely related to moduli spaces of superconformal field theories. For more on this connection see [6].

From a purely mathematical point of view, spaces of stability conditions are interesting because they define new invariants of triangulated categories, and because they provide a way to introduce geometrical ideas into problems of homological algebra. Thus for example, the group $\text{Aut}(\mathcal{D})$ of exact autoequivalences of a triangulated category \mathcal{D} acts on the space $\text{Stab}(\mathcal{D})$ in such a way as to preserve a natural distance function.

The aim of this paper is to give a description of one connected component of the space $\text{Stab}(\mathcal{D})$ in the case when \mathcal{D} is the bounded derived category of coherent sheaves on a complex algebraic K3 surface. The main result is Theorem 1.1 below, which describes a connected component of $\text{Stab}(\mathcal{D})$ and shows how its geometry determines the group $\text{Aut}(\mathcal{D})$, which is at present unknown.

1.1. Suppose then that X is an algebraic K3 surface over \mathbb{C} . Following Mukai [12], one introduces the extended cohomology lattice of X by using the formula

$$((r_1, D_1, s_1), (r_2, D_2, s_2)) = D_1 \cdot D_2 - r_1 s_2 - r_2 s_1$$

to define a symmetric bilinear form on the cohomology ring

$$H^*(X, \mathbb{Z}) = H^0(X, \mathbb{Z}) \oplus H^2(X, \mathbb{Z}) \oplus H^4(X, \mathbb{Z}).$$

The resulting lattice $H^*(X, \mathbb{Z})$ is even and non-degenerate and has signature $(4, 20)$. Let $H^{2,0}(X) \subset H^2(X, \mathbb{C})$ denote the one-dimensional complex subspace spanned by

the class of a nonzero holomorphic two-form Ω on X . An isometry

$$\varphi: H^*(X, \mathbb{Z}) \rightarrow H^*(X, \mathbb{Z})$$

is called a Hodge isometry if $\varphi \otimes \mathbb{C}$ preserves this subspace. The group of Hodge isometries of $H^*(X, \mathbb{Z})$ will be denoted $\text{Aut } H^*(X, \mathbb{Z})$.

Let $\mathcal{D}(X)$ denote the bounded derived category of coherent sheaves on X . The Mukai vector of an object $E \in \mathcal{D}(X)$ is the element of the sublattice

$$\mathcal{N}(X) = \mathbb{Z} \oplus \text{NS}(X) \oplus \mathbb{Z} = H^*(X, \mathbb{Z}) \cap \Omega^\perp \subset H^*(X, \mathbb{C})$$

defined by the formula

$$v(E) = (r(E), c_1(E), s(E)) = \text{ch}(E) \sqrt{\text{td}(X)} \in H^*(X, \mathbb{Z}),$$

where $\text{ch}(E)$ is the Chern character of E and $s(E) = \text{ch}_2(E) + r(E)$.

The Mukai bilinear form makes $\mathcal{N}(X)$ into an even lattice of signature $(2, \rho)$ where $1 \leq \rho \leq 20$ is the Picard number of X . The Riemann-Roch theorem shows that this form is the negative of the Euler form, that is, for any pair of objects E and F of $\mathcal{D}(X)$

$$\chi(E, F) = \sum_i (-1)^i \dim_{\mathbb{C}} \text{Hom}_X^i(E, F) = -(v(E), v(F)).$$

A result of Orlov [14, Proposition 3.5], extending work of Mukai [12, Theorem 4.9], shows that every exact autoequivalence of $\mathcal{D}(X)$ induces a Hodge isometry of the lattice $H^*(X, \mathbb{Z})$. Thus there is a group homomorphism

$$\varpi: \text{Aut } \mathcal{D}(X) \longrightarrow \text{Aut } H^*(X, \mathbb{Z}).$$

The kernel of this homomorphism will be denoted $\text{Aut}^0 \mathcal{D}(X)$.

Examples of exact autoequivalences of $\mathcal{D}(X)$ include twists by line bundles and pullbacks by automorphisms of X . A more interesting class of examples are provided by the twist or reflection functors, first introduced by Mukai [12, Proposition 2.25], and studied in much greater detail by Seidel and Thomas [15]. Recall that an object $E \in \mathcal{D}(X)$ is called spherical if

$$\text{Hom}_X^i(E, E) = \begin{cases} \mathbb{C} & \text{if } i \in \{0, 2\}, \\ 0 & \text{otherwise.} \end{cases}$$

Given a spherical object $E \in \mathcal{D}(X)$, there is a corresponding twist functor $T_E \in \text{Aut } \mathcal{D}(X)$ fitting into the triangle

$$\text{Hom}^\bullet(E, F) \otimes E \xrightarrow{\alpha} F \longrightarrow T_E(F),$$

where the morphism $\alpha: \text{Hom}^\bullet(E, F) \otimes E \rightarrow F$ is the natural evaluation map.

The Riemann-Roch theorem shows that the Mukai vector of a spherical object lies in the root system

$$\Delta(X) = \{\delta \in \mathcal{N}(X) : (\delta, \delta) = -2\}.$$

Conversely it is known that for every $\delta \in \Delta(X)$ there exist (infinitely many) spherical objects with Mukai vector δ . Under the homomorphism ϖ the functor T_E maps to the reflection

$$v \mapsto v + (\delta, v)\delta,$$

where $\delta \in \Delta(X)$ is the Mukai vector of E . Thus the functor T_E^2 defines an element of $\text{Aut}^0 \mathcal{D}(X)$. Since $T_E(E) = E[-1]$, this element has infinite order.

1.2. In order to bring geometrical methods to bear on the problem of computing the group $\text{Aut} \mathcal{D}(X)$ one needs to introduce a space on which it acts. This is provided by the space of stability conditions on $\mathcal{D}(X)$.

The precise definition will be recalled in Section 2 below, but roughly speaking, a stability condition $\sigma = (Z, \mathcal{P})$ on a triangulated category \mathcal{D} consists of a group homomorphism

$$Z: K(\mathcal{D}) \rightarrow \mathbb{C},$$

where $K(\mathcal{D})$ is the Grothendieck group of \mathcal{D} , and a collection of full subcategories $\mathcal{P}(\phi) \subset \mathcal{D}$, one for each $\phi \in \mathbb{R}$, which together satisfy a system of axioms. The map Z is known as the central charge of the stability condition σ , and the objects of the subcategory $\mathcal{P}(\phi)$ are said to be semistable of phase ϕ in σ .

Suppose now that $\mathcal{D} = \mathcal{D}(X)$ is the derived category of a K3 surface. A stability condition $\sigma = (Z, \mathcal{P})$ on $\mathcal{D}(X)$ is said to be numerical if the central charge Z takes the form

$$Z(E) = (\pi(\sigma), v(E))$$

for some vector $\pi(\sigma) \in \mathcal{N}(X) \otimes \mathbb{C}$. It was shown in [5] that the set of numerical stability conditions on $\mathcal{D}(X)$ satisfying a certain technical axiom called local-finiteness form the points of a complex manifold $\text{Stab}(X)$. Furthermore the map

$$\pi: \text{Stab}(X) \rightarrow \mathcal{N}(X) \otimes \mathbb{C}$$

sending a stability condition to the corresponding vector $\pi(\sigma) \in \mathcal{N}(X) \otimes \mathbb{C}$ is continuous. To understand the image of this map π , first define an open subset

$$\mathcal{P}(X) \subset \mathcal{N}(X) \otimes \mathbb{C}$$

consisting of those vectors whose real and imaginary parts span positive definite two-planes in $\mathcal{N}(X) \otimes \mathbb{R}$. This space has two connected components that are exchanged by complex conjugation.

Note that $\mathrm{GL}^+(2, \mathbb{R})$ acts freely on $\mathcal{P}(X)$ by identifying $\mathcal{N}(X) \otimes \mathbb{C}$ with $\mathcal{N}(X) \otimes \mathbb{R}^2$. A section of this action is provided by the submanifold

$$\mathcal{Q}(X) = \{\bar{U} \in \mathcal{P}(X) : (\bar{U}, \bar{U}) = 0, (\bar{U}, \bar{U}) > 0, r(\bar{U}) = 1\} \subset \mathcal{N}(X) \otimes \mathbb{C},$$

where $r(\bar{U})$ denotes the projection of $\bar{U} \in \mathcal{N}(X) \otimes \mathbb{C} \subset H^*(X, \mathbb{C})$ into $H^0(X, \mathbb{C})$. The manifold $\mathcal{Q}(X)$ can be identified with the tube domain

$$\{\beta + i\omega \in \mathrm{NS}(X) \otimes \mathbb{C} : \omega^2 > 0\}$$

via the exponential map

$$\bar{U} = \exp(\beta + i\omega) = \left(1, \beta + i\omega, \frac{1}{2}(\beta^2 - \omega^2) + i(\beta \cdot \omega)\right).$$

Let $\mathcal{P}^+(X) \subset \mathcal{P}(X)$ denote the connected component containing vectors of the form $\exp(\beta + i\omega)$ for ample divisor classes $\omega \in \mathrm{NS}(X) \otimes \mathbb{R}$. For each $\delta \in \Delta(X)$ let

$$\delta^\perp = \{\bar{U} \in \mathcal{N}(X) \otimes \mathbb{C} : (\bar{U}, \delta) = 0\} \subset \mathcal{N}(X) \otimes \mathbb{C}$$

be the corresponding complex hyperplane. The following is the main result of this paper.

Theorem 1.1. *There is a connected component $\mathrm{Stab}^\dagger(X) \subset \mathrm{Stab}(X)$ which is mapped by π onto the open subset*

$$\mathcal{P}_0^+(X) = \mathcal{P}^+(X) \setminus \bigcup_{\delta \in \Delta(X)} \delta^\perp \subset \mathcal{N}(X) \otimes \mathbb{C}.$$

Moreover, the induced map $\pi: \mathrm{Stab}^\dagger(X) \rightarrow \mathcal{P}_0^+(X)$ is a covering map, and the subgroup of $\mathrm{Aut}^0 \mathcal{D}(X)$ which preserves the connected component $\mathrm{Stab}^\dagger(X)$ acts freely on $\mathrm{Stab}^\dagger(X)$ and is the group of deck transformations of π .

The proof of Theorem 1.1 depends on a detailed analysis of the open subset $U(X) \subset \mathrm{Stab}^\dagger(X)$ consisting of stability conditions in which each skyscraper sheaf \mathcal{O}_x is stable of the same phase. It turns out that these stability conditions can be constructed explicitly using the method of abstract tilting first introduced by Happel, Reiten and Smalø [9].

Once one has understood the subset $U(X)$ it is possible to show that the components of its boundary are exchanged by the action of certain autoequivalences of $\mathcal{D}(X)$. Since these autoequivalences reverse the orientation of the boundary this quickly leads to a proof of the statement that every stability condition in $\mathrm{Stab}^\dagger(X)$ is mapped into the closure of $U(X)$ by some autoequivalence $\mathcal{D}(X)$. Theorem 1.1 then follows relatively easily.

1.3. Unfortunately, Theorem 1.1 is not enough to determine the group $\text{Aut } \mathcal{D}(X)$. The argument of Orlov [14, Proposition 3.5] shows that the image of the homomorphism ϖ contains the index two subgroup

$$\text{Aut}^+ H^*(X, \mathbb{Z}) \subset \text{Aut } H^*(X, \mathbb{Z})$$

consisting of elements which do not exchange the two components of $\mathcal{P}(X)$. But as pointed out by Szendrői [16], it is not known whether the map ϖ is onto.

Note that any element of $\text{Aut } \mathcal{D}(X)$ not mapping into $\text{Aut}^+ H^*(X, \mathbb{Z})$ could not preserve the component $\text{Stab}^\dagger(X)$. A related issue is that there may be several components of $\text{Stab}(X)$ which are permuted by the action of $\text{Aut}^0 \mathcal{D}(X)$. Finally the space $\text{Stab}^\dagger(X)$ may not be simply-connected. A proof of the following geometrical statement would solve all these problems.

Conjecture 1.2. *The action of $\text{Aut } \mathcal{D}(X)$ on $\text{Stab}(X)$ preserves the connected component $\text{Stab}^\dagger(X)$. Moreover $\text{Stab}^\dagger(X)$ is simply-connected. Thus there is a short exact sequence of groups*

$$1 \longrightarrow \pi_1 \mathcal{P}_0^+(X) \longrightarrow \text{Aut } \mathcal{D}(X) \xrightarrow{\varpi} \text{Aut}^+ H^*(X, \mathbb{Z}) \longrightarrow 1.$$

It seems reasonable to hope that a more detailed analysis of $\text{Stab}(X)$, and in particular its natural distance function, will lead to a proof of this conjecture.

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2. STABILITY CONDITIONS

The notion of a stability condition on a triangulated category was introduced in [5]. The next three sections contain a summary of the contents of that paper together with a few additional statements that will be needed later on.

Definition 2.1. A stability condition $\sigma = (Z, \mathcal{P})$ on a triangulated category \mathcal{D} consists of a linear map $Z: K(\mathcal{D}) \rightarrow \mathbb{C}$ called the central charge, and full additive subcategories $\mathcal{P}(\phi) \subset \mathcal{D}$ for each $\phi \in \mathbb{R}$, satisfying the following axioms:

- (a) if $0 \neq E \in \mathcal{P}(\phi)$ then $Z(E) = m(E) \exp(i\pi\phi)$ for some $m(E) \in \mathbb{R}_{>0}$,
- (b) for all $\phi \in \mathbb{R}$, $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$,
- (c) if $\phi_1 > \phi_2$ and $A_j \in \mathcal{P}(\phi_j)$ then $\text{Hom}_{\mathcal{D}}(A_1, A_2) = 0$,
- (d) for $0 \neq E \in \mathcal{D}$ there is a finite sequence of real numbers

$$\phi_1 > \phi_2 > \cdots > \phi_n$$

and a collection of triangles

$$\begin{array}{ccccccc}
 0 = E_0 & \longrightarrow & E_1 & \longrightarrow & E_2 & \longrightarrow \cdots \longrightarrow & E_{n-1} & \longrightarrow & E_n = E \\
 & & \swarrow & & \swarrow & & \swarrow & & \swarrow \\
 & & A_1 & & A_2 & & A_n & &
 \end{array}$$

with $A_j \in \mathcal{P}(\phi_j)$ for all j .

Given a stability condition as in the definition, each subcategory $\mathcal{P}(\phi)$ is abelian. The nonzero objects of $\mathcal{P}(\phi)$ are said to be semistable of phase ϕ in σ , and the simple objects of $\mathcal{P}(\phi)$ are said to be stable. It is an easy exercise to check that the decompositions of a nonzero object $0 \neq E \in \mathcal{D}$ given by axiom (d) are uniquely defined up to isomorphism. The objects A_j will be called the semistable factors of E with respect to σ . Write $\phi_\sigma^+(E) = \phi_1$ and $\phi_\sigma^-(E) = \phi_n$; clearly $\phi_\sigma^-(E) \leq \phi_\sigma^+(E)$ with equality holding precisely when E is semistable in σ . The mass of E is defined to be the positive real number $m_\sigma(E) = \sum_i |Z(A_i)|$. By the triangle inequality one has $m_\sigma(E) \geq |Z(E)|$. When the stability condition σ is clear from the context I often drop it from the notation and write $\phi^\pm(E)$ and $m(E)$.

For any interval $I \subset \mathbb{R}$, define $\mathcal{P}(I)$ to be the extension-closed subcategory of \mathcal{D} generated by the subcategories $\mathcal{P}(\phi)$ for $\phi \in I$. Thus, for example, the full subcategory $\mathcal{P}((a, b))$ consists of the zero objects of \mathcal{D} together with those objects $0 \neq E \in \mathcal{D}$ which satisfy $a < \phi^-(E) \leq \phi^+(E) < b$.

To prove nice results it is necessary to put one extra condition on stability conditions. A stability condition is called locally finite if there is some $\epsilon > 0$ such that each quasi-abelian category $\mathcal{P}((\phi - \epsilon, \phi + \epsilon))$ is of finite length. For more details on this see [5]. If $\sigma = (Z, \mathcal{P})$ is locally finite then each subcategory $\mathcal{P}(\phi)$ has finite length, so that every semistable object has a finite Jordan-Hölder filtration into stable factors of the same phase.

The set $\text{Stab}(\mathcal{D})$ of locally-finite stability conditions on a fixed triangulated category \mathcal{D} has a natural topology induced by the generalised metric

$$d(\sigma_1, \sigma_2) = \sup_{0 \neq E \in \mathcal{D}} \left\{ |\phi_{\sigma_2}^-(E) - \phi_{\sigma_1}^-(E)|, |\phi_{\sigma_2}^+(E) - \phi_{\sigma_1}^+(E)|, \left| \log \frac{m_{\sigma_2}(E)}{m_{\sigma_1}(E)} \right| \right\} \in [0, \infty].$$

With this topology the functions $m_\sigma(E)$ and $\phi_\sigma^\pm(E)$ are continuous for every nonzero object $E \in \mathcal{D}$. It follows that the subset of $\text{Stab}(\mathcal{D})$ where a given object $E \in \mathcal{D}$ is semistable is a closed subset.

Lemma 2.2. [5, Lemma 8.2] *The generalised metric space $\text{Stab}(\mathcal{D})$ carries a right action of the group $\text{GL}^+(2, \mathbb{R})$, the universal cover of $\text{GL}^+(2, \mathbb{R})$, and a left action by isometries of the group $\text{Aut}(\mathcal{D})$ of exact autoequivalences of \mathcal{D} . These two actions commute.*

Proof. First note that the group $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ can be thought of as the set of pairs (T, f) where $f: \mathbb{R} \rightarrow \mathbb{R}$ is an increasing map with $f(\phi+1) = f(\phi) + 1$, and $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is an orientation-preserving linear isomorphism, such that the induced maps on $S^1 = \mathbb{R}/2\mathbb{Z} = (\mathbb{R}^2 \setminus \{0\})/\mathbb{R}_{>0}$ are the same.

Given a stability condition $\sigma = (Z, \mathcal{P}) \in \mathrm{Stab}(\mathcal{D})$, and a pair $(T, f) \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$, define a new stability condition $\sigma' = (Z', \mathcal{P}')$ by setting $Z' = T^{-1} \circ Z$ and $\mathcal{P}'(\phi) = \mathcal{P}(f(\phi))$. Note that the semistable objects of the stability conditions σ and σ' are the same, but the phases have been relabelled.

For the second action, note that an element $\Phi \in \mathrm{Aut}(\mathcal{D})$ induces an automorphism ϕ of $K(\mathcal{D})$. If $\sigma = (Z, \mathcal{P})$ is a stability condition on \mathcal{D} , define $\Phi(\sigma)$ to be the stability condition $(Z \circ \phi^{-1}, \mathcal{P}')$, where $\mathcal{P}'(t) = \mathcal{P}(\Phi(t))$. The reader can easily check that this action is by isometries and commutes with the first. \square

If σ and τ are stability conditions on a triangulated category \mathcal{D} define

$$f(\sigma, \tau) = \sup_{0 \neq E \in \mathcal{D}} \left\{ |\phi_{\tau}^{-}(E) - \phi_{\sigma}^{-}(E)|, |\phi_{\tau}^{+}(E) - \phi_{\sigma}^{+}(E)| \right\} \in [0, \infty].$$

For any connected component $\mathrm{Stab}^*(\mathcal{D}) \subset \mathrm{Stab}(\mathcal{D})$ the function $f: \mathrm{Stab}^*(\mathcal{D}) \times \mathrm{Stab}^*(\mathcal{D}) \rightarrow \mathbb{R}_{\geq 0}$ is continuous and finite. It appeared in [5] as a generalised metric on the space of slicings.

Lemma 2.3. [5, Lemma 6.4] *If σ and τ are stability conditions on a triangulated category \mathcal{D} with the same central charge, and $f(\sigma, \tau) < 1$, then $\sigma = \tau$.* \square

The main result of [5] is the following deformation result. The idea is that if $\sigma = (Z, \mathcal{P})$ is a stability condition on a triangulated category \mathcal{D} and one deforms Z to a new group homomorphism $W: K(\mathcal{D}) \rightarrow \mathbb{C}$ in such a way that the phase of each stable object in σ changes in a uniformly bounded way, then it is possible to define new classes of semistable objects $\mathcal{Q}(\psi) \subset \mathcal{D}$ so that (W, \mathcal{Q}) is a stability condition on \mathcal{D} .

Theorem 2.4. [5, Theorem 7.1] *Let $\sigma = (Z, \mathcal{P})$ be a locally-finite stability condition on a triangulated category \mathcal{D} . Take $0 < \epsilon < \frac{1}{8}$ such that each of the quasi-abelian categories $\mathcal{P}((t - 4\epsilon, t + 4\epsilon)) \subset \mathcal{D}$ is of finite length. If $W: K(\mathcal{D}) \rightarrow \mathbb{C}$ is a group homomorphism satisfying*

$$|W(E) - Z(E)| < \sin(\pi\epsilon)|Z(E)|$$

for all objects $E \in \mathcal{D}$ which are stable in σ , then there is a locally-finite stability condition $\tau = (W, \mathcal{Q})$ on \mathcal{D} with $f(\sigma, \tau) < \epsilon$. \square

In fact in [5, Theorem 7.1] the inequality was assumed for all objects $E \in \mathcal{D}$ which are semistable in σ , but it is clear that it is enough to check it for stable

objects, since any semistable object has a finite filtration by stable objects of the same phase.

3. T-STRUCTURES AND STABILITY FUNCTIONS

A stability condition on a triangulated category \mathcal{D} consists of a t-structure on \mathcal{D} together with a stability function on its heart. This statement is made precise in Proposition 3.5 below. Readers unfamiliar with the concept of a t-structure should consult [2, 8]. The following easy result is a good exercise.

Lemma 3.1. *A bounded t-structure is determined by its heart. Moreover, if $\mathcal{A} \subset \mathcal{D}$ is a full additive subcategory of a triangulated category \mathcal{D} , then \mathcal{A} is the heart of a bounded t-structure on \mathcal{D} if and only if the following two conditions hold:*

- (a) *if A and B are objects of \mathcal{A} then $\mathrm{Hom}_{\mathcal{D}}(A, B[k]) = 0$ for $k < 0$,*
- (b) *for every nonzero object $E \in \mathcal{D}$ there are integers $m < n$ and a collection of triangles*

$$\begin{array}{ccccccc}
 0 = E_m & \longrightarrow & E_{m+1} & \longrightarrow & E_{m+2} & \longrightarrow \cdots \longrightarrow & E_{n-1} & \longrightarrow & E_n = E \\
 & & \swarrow & & \swarrow & & \swarrow & & \swarrow \\
 & & A_{m+1} & & A_{m+2} & & A_n & &
 \end{array}$$

with $A_i[i] \in \mathcal{A}$ for all i .

□

In analogy with the standard t-structure on the derived category of an abelian category, the objects $A_i[i]$ of \mathcal{A} are called the cohomology objects of A in the given t-structure, and one often writes $H^i(E) = A_i[i]$.

A very useful method for constructing t-structures is provided by the idea of tilting with respect to a torsion pair, first introduced by D. Happel, I. Reiten and S. Smalø [9].

Definition 3.2. A torsion pair in an abelian category \mathcal{A} is a pair of full subcategories $(\mathcal{T}, \mathcal{F})$ of \mathcal{A} which satisfy $\mathrm{Hom}_{\mathcal{A}}(T, F) = 0$ for $T \in \mathcal{T}$ and $F \in \mathcal{F}$, and such that every object $E \in \mathcal{A}$ fits into a short exact sequence

$$0 \longrightarrow T \longrightarrow E \longrightarrow F \longrightarrow 0$$

for some pair of objects $T \in \mathcal{T}$ and $F \in \mathcal{F}$.

The objects of \mathcal{T} and \mathcal{F} are called torsion and torsion-free respectively. The following Lemma is pretty-much immediate from Lemma 3.1.

Lemma 3.3. [9, Proposition 2.1] *Suppose \mathcal{A} is the heart of a bounded t-structure on a triangulated category \mathcal{D} . Given an object $E \in \mathcal{D}$ let $H^i(E) \in \mathcal{A}$ denote the i th cohomology object of E with respect to this t-structure. Suppose $(\mathcal{T}, \mathcal{F})$ is a torsion pair in \mathcal{A} . Then the full subcategory*

$$\mathcal{A}^\sharp = \{E \in \mathcal{D} : H^i(E) = 0 \text{ for } i \notin \{-1, 0\}, H^{-1}(E) \in \mathcal{F} \text{ and } H^0(E) \in \mathcal{T}\}$$

is the heart of a bounded t -structure on \mathcal{D} . \square

One says that \mathcal{A}^\sharp is obtained from the category \mathcal{A} by tilting with respect to the torsion pair $(\mathcal{T}, \mathcal{F})$. Note that the pair $(\mathcal{F}[1], \mathcal{T})$ is a torsion pair in \mathcal{A}^\sharp and that tilting with respect to this pair gives back the original category \mathcal{A} with a shift.

Definition 3.4. A stability function on an abelian category \mathcal{A} is a group homomorphism $Z: K(\mathcal{A}) \rightarrow \mathbb{C}$ such that

$$0 \neq E \in \mathcal{A} \implies Z(E) \in \mathbb{R}_{>0} \exp(i\pi\phi(E)) \text{ with } 0 < \phi(E) \leq 1.$$

The real number $\phi(E) \in (0, 1]$ is called the phase of the object E .

A nonzero object $E \in \mathcal{A}$ is said to be semistable with respect to a stability function Z if

$$0 \neq A \subset E \implies \phi(A) \leq \phi(E).$$

The stability function Z is said to have the Harder-Narasimhan property if every nonzero object $E \in \mathcal{A}$ has a finite filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = E$$

whose factors $F_j = E_j/E_{j-1}$ are semistable objects of \mathcal{A} with

$$\phi(F_1) > \phi(F_2) > \cdots > \phi(F_n).$$

A simple sufficient condition for the existence of Harder-Narasimhan filtrations was given in [5, Proposition 2.4].

Proposition 3.5. [5, Proposition 5.3] *To give a stability condition on a triangulated category \mathcal{D} is equivalent to giving a bounded t -structure on \mathcal{D} and a stability function on its heart which has the Harder-Narasimhan property.*

Proof. If $\sigma = (Z, \mathcal{P})$ is a stability condition on \mathcal{D} , the abelian subcategory $\mathcal{A} = \mathcal{P}((0, 1])$ is the heart of a bounded t -structure on \mathcal{D} . The central charge Z defines a stability function on \mathcal{A} and it is easy to check that the corresponding semistable objects are precisely the nonzero objects of the subcategories $\mathcal{P}(\phi)$ for $0 < \phi \leq 1$. The decompositions of objects of \mathcal{A} given by Definition 2.1(d) are Harder-Narasimhan filtrations.

For the converse, suppose \mathcal{A} is the heart of a bounded t -structure on \mathcal{D} , and $Z: K(\mathcal{A}) \rightarrow \mathbb{C}$ is a stability function on \mathcal{A} with the Harder-Narasimhan property. Define a stability condition $\sigma = (Z, \mathcal{P})$ on \mathcal{D} as follows. For each $\phi \in (0, 1]$ let $\mathcal{P}(\phi)$ be the full additive subcategory of \mathcal{D} consisting of semistable objects of \mathcal{A} with phase ϕ , together with the zero objects of \mathcal{D} . Condition (b) of Definition 2.1 then determines $\mathcal{P}(\phi)$ for all $\phi \in \mathbb{R}$ and conditions (a) and (c) are easily verified. Given a nonzero object $E \in \mathcal{D}$, a filtration as in Definition 2.1(d) can be obtained by

combining the decompositions of Lemma 3.1 with the Harder-Narasimhan filtrations of the cohomology objects of E . \square

In Section 6 this result will be combined with the method of tilting and used to construct examples of stability conditions on K3 surfaces.

4. STABILITY CONDITIONS ON VARIETIES

In this paper the triangulated category \mathcal{D} will always be the bounded derived category of coherent sheaves $\mathcal{D}(X)$ on a smooth projective variety X over the complex numbers. Any such category is of finite type, that is, for any pair of objects E and F in \mathcal{D} the morphism space $\bigoplus_i \mathrm{Hom}_X(E, F[i])$ is a finite-dimensional vector space over \mathbb{C} . The Euler characteristic

$$\chi(E, F) = \sum_i (-1)^i \dim_{\mathbb{C}} \mathrm{Hom}_X(E, F[i]),$$

then defines a bilinear form on the Grothendieck group $K(X)$. Serre duality shows that the left- and right-radicals $K(X)^\perp$ and ${}^\perp K(X)$ are the same, so that the Euler form descends to a nondegenerate form $\chi(-, -)$ on the numerical Grothendieck group

$$\mathcal{N}(X) = K(X)/K(X)^\perp.$$

The Riemann-Roch theorem shows that this free abelian group has finite rank. A stability condition $\sigma = (Z, \mathcal{P})$ is said to be numerical if the central charge $Z: K(X) \rightarrow \mathbb{C}$ factors through the quotient group $\mathcal{N}(X)$. An equivalent condition is that the central charge Z takes the form

$$Z(E) = -\chi(\pi(\sigma), E)$$

for some vector $\pi(\sigma) \in \mathcal{N}(X) \otimes \mathbb{C}$. Here E is simultaneously representing an element of $K(X)$ and the quotient $\mathcal{N}(X)$. The sign is just to ensure agreement with later conventions.

The set of all locally finite numerical stability conditions on $\mathcal{D}(X)$ with its natural topology will be denoted $\mathrm{Stab}(X)$. The group $\mathrm{Aut} \mathcal{D}(X)$ of exact autoequivalences of $\mathcal{D}(X)$ acts on $\mathrm{Stab}(X)$. Theorem 2.4 leads to the following result.

Theorem 4.1. [5, Cor. 1.3] *For each connected component $\mathrm{Stab}^*(X) \subset \mathrm{Stab}(X)$ there is a linear subspace $V \subset \mathcal{N}(X) \otimes \mathbb{C}$ such that*

$$\pi: \mathrm{Stab}^*(X) \longrightarrow \mathcal{N}(X) \otimes \mathbb{C}$$

is a local homeomorphism onto an open subset of the subspace V . In particular $\mathrm{Stab}^(X)$ is a finite-dimensional complex manifold.*

In all known examples, the subspace V of Theorem 4.1 is actually equal to $\mathcal{N}(X) \otimes \mathbb{C}$. Since it is not known whether this is always the case the following definition will be useful.

Definition 4.2. A connected component $\text{Stab}^*(X) \subset \text{Stab}(X)$ will be called full if the subspace V of Theorem 4.1 is equal to $\mathcal{N}(X) \otimes \mathbb{C}$. A stability condition $\sigma \in \text{Stab}(X)$ is full if it lies in a full component.

Note that if a stability condition $\sigma \in \text{Stab}(X)$ is full and $\Phi \in \text{Aut } \mathcal{D}(X)$ is an autoequivalence then the stability condition $\Phi(\sigma)$ is also full.

Later on it will be important to be able to choose the constant ϵ appearing in Theorem 2.4 uniformly. This can be done for full stability conditions. To do so one first considers discrete stability conditions.

Definition 4.3. A stability condition $\sigma = (Z, \mathcal{P})$ on \mathcal{D} is called discrete if the image of $Z: K(\mathcal{D}) \rightarrow \mathbb{C}$ is a discrete subgroup.

Lemma 4.4. *Suppose $\sigma = (Z, \mathcal{P})$ is a discrete stability condition and fix $0 < \epsilon < \frac{1}{2}$. Then for each $\phi \in \mathbb{R}$ the quasi-abelian category $\mathcal{P}((\phi - \epsilon, \phi + \epsilon))$ is of finite length. In particular σ is locally finite.*

Proof. Fix $\phi \in \mathbb{R}$ and set $\mathcal{A} = \mathcal{P}((\phi - \epsilon, \phi + \epsilon))$. The central charge of any nonzero object $A \in \mathcal{A}$ lies in the sector

$$S = \{z = r \exp(i\pi\psi) : r > 0 \text{ and } \phi - \epsilon < \psi < \phi + \epsilon\}$$

which is strictly smaller than a half-plane in \mathbb{C} . Set $f(A) = \text{Re}(\exp(-i\pi\phi)Z(A))$. Then $f(A) > 0$ for all nonzero objects $A \in \mathcal{A}$. If

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is a strict short exact sequence in \mathcal{A} then $f(B) = f(A) + f(C)$. Thus for a given object $E \in \mathcal{A}$ the central charges of all sub and quotient objects lie in the bounded region

$$\{z \in S : \text{Re}(\exp(-i\pi\phi)z) < f(E)\}$$

Since σ is discrete there are only finitely many possibilities, and so any chains of sub or quotient objects must terminate. \square

Lemma 4.5. *Suppose $\sigma = (Z, \mathcal{P}) \in \text{Stab}(X)$ is a full stability condition and fix $0 < \epsilon < \frac{1}{2}$. Then for each $\phi \in \mathbb{R}$ the quasi-abelian category $\mathcal{P}((\phi - \epsilon, \phi + \epsilon))$ is of finite length.*

Proof. The basic point is that there exist discrete stability conditions in $\text{Stab}(X)$ arbitrarily close to σ . Indeed, one can approximate $\pi(\sigma) \in \mathcal{N}(X) \otimes \mathbb{C}$ arbitrarily closely by points in $\mathcal{N}(X) \otimes \mathbb{Q}[i]$. By the fullness assumption these points can be lifted to stability conditions $\tau = (W, \mathcal{Q})$ lying arbitrarily close to σ . The resulting stability conditions will clearly be discrete. But by [5, Lemma 6.1], if $f(\sigma, \tau) < \eta$ then $\mathcal{P}((\phi - \epsilon, \phi + \epsilon)) \subset \mathcal{Q}((\phi - \epsilon - \eta, \phi + \epsilon + \eta))$ so the result follows from Lemma 4.4. \square

5. SHEAVES ON K3 SURFACES

This section contains some basic results about coherent sheaves on K3 surfaces. These are mostly taken from Mukai's excellent paper [12]. Throughout notation will be as in the introduction. In particular X is a fixed algebraic K3 surface over \mathbb{C} and $\mathcal{D}(X)$ is the corresponding bounded derived category of coherent sheaves.

There are various notions of stability for sheaves on a K3 surface, the most basic of which is slope-stability. Recall that if $\omega \in \text{NS}(X)$ is an ample divisor class, one defines the slope $\mu_\omega(E)$ of a torsion-free sheaf E on X to be the quotient

$$\mu_\omega(E) = \frac{c_1(E) \cdot \omega}{r(E)}.$$

A torsion-free sheaf E is said to be μ_ω -semistable if $\mu_\omega(A) \leq \mu_\omega(E)$ for every subsheaf $0 \neq A \subset E$. If the inequality is always strict when A is of strictly smaller rank then E is said to be μ_ω -stable. This definition can be extended to include any class ω in the ample cone of X ,

$$\text{Amp}(X) = \{\omega \in \text{NS}(X) \otimes \mathbb{R} : \omega^2 > 0 \text{ and } \omega \cdot C > 0 \text{ for any curve } C \subset X\}.$$

One very important point to realise is that slope-stability does not arise from a stability condition on $\mathcal{D}(X)$. The function $Z(E) = -c_1(E) \cdot \omega + ir(E)$ is not a stability function on the category of coherent sheaves on X , because it is zero on any sheaf supported in dimension zero. Thus constructing examples of stability conditions on $\mathcal{D}(X)$ is a non-trivial problem, which will be tackled in the next section.

Sending an object $E \in \mathcal{D}(X)$ to its Mukai vector

$$v(E) = (r(E), c_1(E), s(E)) = \text{ch}(E)\sqrt{\text{td}(X)} \in H^*(X, \mathbb{Z})$$

identifies the numerical Grothendieck group of X with the sublattice

$$\mathcal{N}(X) = \mathbb{Z} \oplus \text{NS}(X) \oplus \mathbb{Z} = H^*(X, \mathbb{Z}) \cap \Omega^\perp \subset H^*(X, \mathbb{C})$$

defined in the introduction. The Riemann-Roch theorem then shows that the Mukai bilinear form induces the negative of the Euler form on $\mathcal{N}(X)$ [12, Proposition 2.2],

so that for any pair of objects E and F of $\mathcal{D}(X)$

$$\chi(E, F) = \sum_i (-1)^i \dim_{\mathbb{C}} \operatorname{Hom}_X^i(E, F) = -(v(E), v(F)).$$

Recall also [12, Proposition 2.3] that Serre duality gives isomorphisms

$$\operatorname{Hom}_X^i(E, F) \cong \operatorname{Hom}_X^{2-i}(F, E)^*.$$

If the objects E and F lie in the heart of some t-structure on $\mathcal{D}(X)$ these spaces vanish for $i < 0$ and hence also for $i > 2$. This is the case for example when E and F are both sheaves, or if E and F are semistable of the same phase in some stability condition. In this situation, combining Riemann-Roch and Serre duality often allows one to determine the spaces $\operatorname{Hom}_X^i(E, F)$. The next Lemma is a good example.

Lemma 5.1. [12, Corollary 2.5] *If $E \in \mathcal{D}(X)$ is stable in some stability condition on X , or is a μ_ω -stable sheaf for some $\omega \in \operatorname{Amp}(X)$, then*

$$\dim_{\mathbb{C}} \operatorname{Hom}_X^1(E, E) = 2 + v(E)^2 \geq 0,$$

with equality precisely when E is spherical.

Proof. Any stable object satisfies $\operatorname{Hom}_X(E, E) = \mathbb{C}$, so that by Serre duality, one has $\operatorname{Hom}_X^2(E, E) = \mathbb{C}$ also, and the given inequality follows from Riemann-Roch. \square

The following Lemma is essentially due to Mukai.

Lemma 5.2. [12, Corollary 2.8] *Suppose $\mathcal{A} \subset \mathcal{D}(X)$ is the heart of a bounded t-structure and*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is a short exact sequence in \mathcal{A} with $\operatorname{Hom}_X(A, C) = 0$. Then

$$\dim_{\mathbb{C}} \operatorname{Hom}_X^1(A, A) + \dim_{\mathbb{C}} \operatorname{Hom}_X^1(C, C) \leq \dim_{\mathbb{C}} \operatorname{Hom}_X^1(B, B).$$

Proof. Given objects E and F of \mathcal{A} write

$$(E, F)^i = \dim_{\mathbb{C}} \operatorname{Hom}_X^i(E, F).$$

Thus $(E, F)^i = 0$ unless $0 \leq i \leq 2$, and Serre duality gives $(E, F)^i = (F, E)^{2-i}$.

The stated inequality is equivalent to

$$(A, A)^0 + (C, C)^0 + (C, A)^0 \leq (C, A)^1 + (B, B)^0,$$

which follows from the existence of an exact sequence

$$0 \rightarrow \operatorname{Hom}_X(C, A) \rightarrow \operatorname{End}_X(B) \rightarrow \operatorname{End}_X(A) \oplus \operatorname{End}_X(C) \rightarrow \operatorname{Hom}_X^1(C, A).$$

To see where this sequence comes from note that because $(A, C)^0 = 0$, any endomorphism of B induces an endomorphism of the whole triangle $A \rightarrow B \rightarrow C$, which

is just a pair of endomorphisms of A and C preserving the class of the connecting morphism $C \rightarrow A[1]$. \square

The following important result of Yoshioka gives existence of semistable sheaves.

Theorem 5.3. (Yoshioka) *Suppose $v \in \mathcal{N}(X)$ satisfies $v^2 \geq -2$ and $r(v) > 0$. Then for any ample divisor class $\omega \in \text{Amp}(X)$, there are torsion-free μ_ω -semistable sheaves on X with Mukai vector v .*

Proof. Since one is looking only for semistable sheaves it is enough to check the case when $v \in \mathcal{N}(X)$ is primitive. Using the wall and chamber structure of $\text{Amp}(X)$ (see for example [17]) one can also deform ω a little so that it is general for v , in the sense that all μ_ω -semistable sheaves with Mukai vector v are actually μ_ω -stable. The result then follows from [18, Theorem 8.1]. \square

6. CONSTRUCTING STABILITY CONDITIONS

Having dealt with the preliminaries it is now possible to make a start on the proof of Theorem 1.1. For the next nine sections X will be a fixed algebraic K3 surface over \mathbb{C} . In the next two sections we use the method of tilting to construct examples of stability conditions on the bounded derived category of coherent sheaves $\mathcal{D}(X)$.

Take a pair of \mathbb{R} -divisors $\beta, \omega \in \text{NS}(X) \otimes \mathbb{R}$ such that $\omega \in \text{Amp}(X)$ lies in the ample cone. As in the introduction, define a group homomorphism $Z: \mathcal{N}(X) \rightarrow \mathbb{C}$ by the formula

$$Z(E) = (\exp(\beta + i\omega), v(E)).$$

A little rewriting shows that if $E \in \mathcal{D}(X)$ has Mukai vector (r, Δ, s) with $r \neq 0$ then

$$(\star) \quad Z(E) = \frac{1}{2r} \left((\Delta^2 - 2rs) + r^2\omega^2 - (\Delta - r\beta)^2 \right) + i(\Delta - r\beta) \cdot \omega$$

which reduces to $Z(E) = (\Delta \cdot \beta - s) + i(\Delta \cdot \omega)$ when $r = 0$.

Every torsion-free sheaf E on X has a unique Harder-Narasimhan filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = E$$

whose factors $F_i = E_i/E_{i-1}$ are μ_ω -semistable torsion-free sheaves with descending slope μ_ω . Truncating Harder-Narasimhan filtrations at the point $\mu_\omega = \beta \cdot \omega$ leads to the following statement.

Lemma 6.1. *For any pair $\beta, \omega \in \text{NS}(X) \otimes \mathbb{R}$ with $\omega \in \text{Amp}(X)$ there is a unique torsion pair $(\mathcal{T}, \mathcal{F})$ on the category $\text{Coh}(X)$ such that \mathcal{T} consists of sheaves whose torsion-free parts have μ_ω -semistable Harder-Narasimhan factors of slope $\mu_\omega > \beta \cdot \omega$ and \mathcal{F} consists of torsion-free sheaves on X all of whose μ_ω -semistable Harder-Narasimhan factors have slope $\mu_\omega \leq \beta \cdot \omega$. \square*

Tilting with respect to the torsion pair of Lemma 6.1 gives a bounded t-structure on $\mathcal{D}(X)$ with heart

$$\mathcal{A}(\beta, \omega) = \{E \in \mathcal{D}(X) : H^i(E) = 0 \text{ for } i \notin \{-1, 0\}, H^{-1}(E) \in \mathcal{F} \text{ and } H^0(E) \in \mathcal{T}\}.$$

Note that $\mathcal{A}(\beta, \omega)$ does not really depend on β , only on $\beta \cdot \omega$. Note also that all torsion sheaves on X are objects of \mathcal{T} and hence also of the abelian category $\mathcal{A}(\beta, \omega)$.

Lemma 6.2. *Take a pair $\beta, \omega \in \text{NS}(X) \otimes \mathbb{R}$ with $\omega \in \text{Amp}(X)$. Then the group homomorphism Z defined above is a stability function on the abelian category $\mathcal{A}(\beta, \omega)$ providing β and ω are chosen so that for all spherical sheaves E on X one has $Z(E) \notin \mathbb{R}_{\leq 0}$. This holds in particular whenever $\omega^2 > 2$.*

Proof. It is clear that $Z(E)$ lies in the upper half-plane for every sheaf supported on a curve, and every torsion-free semistable sheaf E with $\mu_\omega(E) > \beta \cdot \omega$. Moreover, if E is supported in dimension zero then $Z(E) \in \mathbb{R}_{< 0}$. Similarly, if E is torsion-free with $\mu_\omega(E) < \beta \cdot \omega$ then $Z(E[1])$ lies in the upper half-plane.

The only non-trivial part is to check that if a torsion-free μ_ω -semistable sheaf E satisfies $(\Delta - r\beta) \cdot \omega = 0$ then $Z(E) \in \mathbb{R}_{> 0}$. It is enough to check this when E is μ_ω -stable. By Lemma 5.1, the Mukai vector $v(E) = (r, \Delta, s)$ satisfies

$$\Delta^2 - 2rs = v(E)^2 \geq -2$$

with equality precisely when E is spherical. Since $(\Delta - r\beta) \cdot \omega = 0$ the Hodge index theorem gives $(\Delta - r\beta)^2 \leq 0$, so if E is not spherical then $Z(E)$ lies on the positive real axis. If $\omega^2 > 2$ then $Z(E)$ lies on the positive real axis even if E is spherical. \square

Suppose the pair $\beta, \omega \in \text{NS}(X) \otimes \mathbb{R}$ satisfy the condition of Lemma 6.2. If the corresponding stability function Z has the Harder-Narasimhan property, Proposition 3.5 shows that there is a uniquely-defined stability condition $\sigma = (Z, \mathcal{P})$ on $\mathcal{D}(X)$ with heart $\mathcal{P}((0, 1]) = \mathcal{A}(\beta, \omega)$ and central charge Z . In Section 11 it will be shown that in fact this Harder-Narasimhan property always holds. In the next section a weaker result will be proved, namely that the Harder-Narasimhan property holds when β and ω are rational.

The following observation will be used later to characterise the potential stability conditions constructed above.

Lemma 6.3. *For any pair $\beta, \omega \in \text{NS}(X) \otimes \mathbb{R}$ with $\omega \in \text{Amp}(X)$, and any point $x \in X$, the skyscraper sheaf \mathcal{O}_x is a simple object of the abelian category $\mathcal{A}(\beta, \omega)$.*

Proof. As remarked above, any torsion sheaf on X lies in the torsion subcategory \mathcal{T} and hence in the abelian category $\mathcal{A}(\beta, \omega)$. Suppose

$$0 \longrightarrow A \longrightarrow \mathcal{O}_x \longrightarrow B \longrightarrow 0$$

is a short exact sequence in $\mathcal{A}(\beta, \omega)$. Taking cohomology gives an exact sequence of sheaves on X

$$0 \longrightarrow H^{-1}(B) \longrightarrow H^0(A) \longrightarrow \mathcal{O}_x \longrightarrow H^0(B) \longrightarrow 0.$$

Note that $H^{-1}(B)$ is torsion-free. It follows that the μ_ω -semistable factors of $H^{-1}(B)$ and $H^0(A)$ have the same slope. This contradicts the definition of the category $\mathcal{A}(\beta, \omega)$ unless $H^{-1}(B) = 0$, in which case either A or B must be zero. \square

7. THE HARDER-NARASIMHAN PROPERTY

This section fills a gap in the original version of this paper. I am very grateful to D. Huybrechts for pointing out the error, and to E. Macri and P. Stellari who independently obtained Proposition 7.1 below.

Take $\beta, \omega \in \text{NS}(X) \otimes \mathbb{R}$ with $\omega \in \text{Amp}(X)$ and let $Z: \mathcal{N}(X) \rightarrow \mathbb{C}$ be the group homomorphism

$$Z(E) = (\exp(\beta + i\omega), v(E))$$

defined in the previous section. Suppose that for all spherical sheaves E on X one has $Z(E) \notin \mathbb{R}_{\leq 0}$. According to Lemma 6.2 the map Z then defines a stability function on the abelian category $\mathcal{A}(\beta, \omega) \subset \mathcal{D}(X)$.

Proposition 7.1. *If $\beta, \omega \in \text{NS}(X) \otimes \mathbb{Q}$ are rational then the stability function Z on $\mathcal{A}(\beta, \omega)$ has the Harder-Narasimhan property. The resulting stability conditions on $\mathcal{D}(X)$ are locally-finite.*

Proof. The idea is to apply the criterion of [5, Proposition 2.4]. Note that since β and ω are rational, the image of $Z: \mathcal{N}(X) \rightarrow \mathbb{C}$ is a discrete subgroup of \mathbb{C} . Suppose one has a chain of monomorphisms in $\mathcal{A} = \mathcal{A}(\beta, \omega)$

$$\cdots \subset E_{i+1} \subset E_i \subset \cdots \subset E_1 \subset E_0 = E$$

with increasing phases $\phi(E_{i+1}) > \phi(E_i)$ for all i . There are short exact sequences

$$0 \longrightarrow E_{i+1} \longrightarrow E_i \longrightarrow F_i \longrightarrow 0$$

in \mathcal{A} . Since $\text{Im } Z(A)$ is non-negative for all objects $A \in \mathcal{A}$ one has

$$\text{Im } Z(E_{i+1}) < \text{Im } Z(E_i) \text{ for all } i.$$

Since the image of Z is discrete it follows that for large enough i the value of $\text{Im } Z(E_i)$ is constant. But then $\text{Im } Z(F_i) = 0$ so $\text{Re } Z(F_i) \leq 0$ and so $\text{Re } Z(E_{i+1}) \geq \text{Re } Z(E_i)$. But this contradicts $\phi(E_{i+1}) > \phi(E_i)$. Thus no such chain can exist.

Now suppose one has a chain of epimorphisms

$$E = E_0 \twoheadrightarrow E_1 \twoheadrightarrow \cdots \twoheadrightarrow E_i \twoheadrightarrow E_{i+1} \twoheadrightarrow \cdots$$

with decreasing phases $\phi(E_i) > \phi(E_{i+1})$ for all i . There are short exact sequences

$$0 \longrightarrow K_i \longrightarrow E_i \longrightarrow E_{i+1} \longrightarrow 0.$$

Applying the same argument as for the first part, and omitting a finite number of terms, one can assume that $\text{Im } Z(E_i) = \text{Im } Z(E_{i+1})$ for all i . This time however this does not contradict the inequality on the phases.

Taking long exact sequences in cohomology sheaves shows that there are epimorphisms of sheaves

$$H^0(E_0) \twoheadrightarrow H^0(E_1) \twoheadrightarrow \cdots \twoheadrightarrow H^0(E_i) \twoheadrightarrow H^0(E_{i+1}) \twoheadrightarrow \cdots .$$

Since the category of coherent sheaves is Noetherian this chain must terminate, and so, again omitting a finite number of terms, one can assume that the map $H^0(E) \rightarrow H^0(E_i)$ is an isomorphism for all i .

Consider the composite maps $E = E_0 \rightarrow E_i$ fitting into short exact sequences

$$0 \longrightarrow L_i \longrightarrow E \longrightarrow E_i \longrightarrow 0.$$

Then there is a chain

$$0 = L_0 \subset L_1 \subset \cdots \subset L_i \subset \cdots \subset E$$

and each $\text{Im } Z(L_i) = 0$ for all i . It will be enough to show that for large enough i one has $L_i = L_{i+1}$.

There are short exact sequences

$$0 \longrightarrow L_{i-1} \longrightarrow L_i \longrightarrow B_i \longrightarrow 0.$$

Taking cohomology sheaves shows that there are monomorphisms of sheaves

$$0 = H^{-1}(L_0) \subset H^{-1}(L_1) \subset \cdots \subset H^{-1}(L_i) \subset \cdots \subset H^{-1}(E).$$

Once again, this chain must terminate, so omitting a finite number of terms one can assume that the inclusion $H^{-1}(L_i) \rightarrow H^{-1}(L_{i+1})$ is an isomorphism for all i .

For each i there is a short exact sequence in \mathcal{A}

$$0 \longrightarrow H^{-1}(L_i)[1] \longrightarrow L_i \longrightarrow H^0(L_i) \longrightarrow 0.$$

Since $\text{Im } Z(L_i) = 0$ one must also have $\text{Im } Z(H^0(L_i)) = 0$. But by definition of the category \mathcal{A} this is only possible if $H^0(L_i)$ is a torsion sheaf supported in dimension zero. Taking cohomology gives a long exact sequence of sheaves

$$0 \longrightarrow H^{-1}(B_i) \longrightarrow H^0(L_{i-1}) \longrightarrow H^0(L_i) \longrightarrow H^0(B_i) \longrightarrow 0.$$

Since $H^{-1}(A)$ is torsion-free for any object $A \in \mathcal{A}$, and $H^0(L_{i-1})$ is torsion, one has $H^{-1}(B_i) = 0$.

Now it will be enough to show that $H^0(B_i) = 0$ for large i . Equivalently one must bound the length of the finite-length sheaves $H^0(L_i)$. Consider again the short exact sequence

$$0 \longrightarrow L_i \longrightarrow E \longrightarrow E_i \longrightarrow 0.$$

Taking cohomology sheaves and recalling the assumption that the map $H^0(E) \rightarrow H^0(E_i)$ is an isomorphism for all i gives long exact sequences of sheaves

$$0 \longrightarrow H^{-1}(L_i) \longrightarrow H^{-1}(E) \xrightarrow{f} H^{-1}(E_i) \longrightarrow H^0(L_i) \longrightarrow 0.$$

Let Q be the image of the map f . This is independent of i up to isomorphism since the map $H^{-1}(L_i) \rightarrow H^{-1}(L_{i+1})$ is an isomorphism for all i . But now there is a short exact sequence of sheaves

$$0 \longrightarrow Q \longrightarrow H^{-1}(E_i) \longrightarrow H^0(L_i) \longrightarrow 0$$

in which the middle term $H^{-1}(E_i)$ is torsion-free by definition of the category \mathcal{A} . It follows that Q is torsion-free and $H^0(L_i)$ is a subsheaf of the finite-length sheaf Q^{**}/Q . In particular the length of $H^0(L_i)$ is bounded. This shows that $B_i = 0$ for large i , and hence $E_i = E_{i+1}$ for large i , which contradicts the assumption of decreasing phase.

Finally, the local-finiteness condition follows from Lemma 4.4 because the image of the homomorphism Z is a discrete subgroup of \mathbb{C} . \square

8. THE COVERING MAP PROPERTY

Recall from the introduction that the subset $\mathcal{P}(X) \subset \mathcal{N}(X) \otimes \mathbb{C}$ is defined to be the set of vectors whose real and imaginary parts span positive definite two-planes in $\mathcal{N}(X) \otimes \mathbb{R}$. Recall also that

$$\Delta(X) = \{\delta \in \mathcal{N}(X) : (\delta, \delta) = -2\},$$

and that for each $\delta \in \Delta(X)$ there is a corresponding complex hyperplane

$$\delta^\perp = \{\mathcal{U} \in \mathcal{N}(X) \otimes \mathbb{C} : (\mathcal{U}, \delta) = 0\} \subset \mathcal{N}(X) \otimes \mathbb{C}.$$

The hyperplane complement

$$\mathcal{P}_0(X) = \mathcal{P}(X) \setminus \bigcup_{\delta \in \Delta(X)} \delta^\perp.$$

is an open subset of $\mathcal{P}(X)$ (see Proposition 8.3 below). The aim of this section is to use the deformation result of [5] to show that the map

$$\pi: \text{Stab}(X) \longrightarrow \mathcal{N}(X) \otimes \mathbb{C}$$

is a covering map over this subset.

Recall that a continuous map of topological spaces $f: S \rightarrow T$ is a covering map if every $t \in T$ has an open neighbourhood $t \in V \subset T$ such that the restriction of f to each connected component of $f^{-1}(V) \subset S$ is a homeomorphism onto V . It is an immediate consequence that if S is non-empty and T is connected then f is surjective.

Lemma 8.1. *Let $\|\cdot\|$ be a norm on the finite dimensional vector space $\mathcal{N}(X) \otimes \mathbb{C}$ and let $(-, -)$ denote the Mukai bilinear form on $\mathcal{N}(X) \otimes \mathbb{C}$. Take a vector $\mathcal{U} \in \mathcal{P}(X)$. Then there is a constant $r > 0$ (depending on \mathcal{U}) such that*

$$|(u, v)| \leq r \|u\| |(\mathcal{U}, v)|$$

for all $u \in \mathcal{N}(X) \otimes \mathbb{C}$ and all $v \in \mathcal{N}(X) \otimes \mathbb{R}$ with $(v, v) \geq 0$. If moreover $\mathcal{U} \in \mathcal{P}_0(X)$ then one can choose $r > 0$ so that the same inequality holds for all $u \in \mathcal{N}(X) \otimes \mathbb{C}$ and all $v \in \Delta(X)$.

Proof. The assumption $\mathcal{U} \in \mathcal{P}(X)$ means that there is a basis e_1, \dots, e_n of the real vector space $\mathcal{N}(X) \otimes \mathbb{R}$ which is orthogonal with respect to the Mukai inner product with signature $(+1, +1, -1, \dots, -1)$, and such that the real and imaginary parts of \mathcal{U} are a basis for the subspace spanned by e_1 and e_2 . Since all norms on a finite-dimensional vector space are equivalent, one might as well take

$$\|v\|^2 = v_1^2 + v_2^2 + \dots + v_n^2,$$

where v_i denotes the i -th component of a vector $v \in \mathcal{N}(X) \otimes \mathbb{R}$ with respect to the basis e_1, \dots, e_n . Note that for vectors $u, v \in \mathcal{N}(X) \otimes \mathbb{R}$, the Cauchy-Schwartz inequality gives

$$|(u, v)| = |u_1 v_1 + u_2 v_2 - u_3 v_3 - \dots - u_n v_n| \leq \|u\| \|v\|.$$

Thus it will be enough to show that for some $r > 0$ one has $\|v\| \leq r |(\mathcal{U}, v)|$ for all the given $v \in \mathcal{N}(X) \otimes \mathbb{R}$.

Since the real and imaginary parts of \mathcal{U} are a basis for the subspace spanned by e_1 and e_2 , there is a $k > 0$ such that for all $v \in \mathcal{N}(X) \otimes \mathbb{R}$ one has

$$v_1^2 + v_2^2 \leq k |(\mathcal{U}, v)|^2.$$

If a vector $v \in \mathcal{N}(X) \otimes \mathbb{R}$ satisfies $v^2 \geq 0$ then $\|v\|^2 \leq 2(v_1^2 + v_2^2)$ so the result follows.

In the case when $v \in \Delta(X)$ one has

$$\|v\|^2 = 2(v_1^2 + v_2^2 + 1),$$

so it will be enough to check that

$$1 \leq m |(\mathcal{U}, v)|^2$$

for some $m > 0$. In other words, as v varies in $\Delta(X)$, the quantity $|(\mathcal{U}, v)|$ is bounded below by a positive real number. If $|(\mathcal{U}, v)| \leq 1$ then $\|v\|^2 \leq 2(k+1)$ and v lies in a bounded subset of $\mathcal{N}(X) \otimes \mathbb{R}$. But there are only finitely many integral points in any bounded subset of $\mathcal{N}(X) \otimes \mathbb{R}$, and the assumption $\mathcal{U} \in \mathcal{P}_0(X)$ implies that all the $|(\mathcal{U}, v)|$ are nonzero, so the result follows. \square

It is worth recording the following consequence of the above proof.

Lemma 8.2. *If $\mathcal{U} \in \mathcal{P}_0(X)$ and $m > 0$ is a constant then there are only finitely many elements $v \in \mathcal{N}(X)$ such that $v^2 \geq -2$ and $|(\mathcal{U}, v)| \leq m$.*

Proof. The inequalities given in the proof of Lemma 8.1 give an upper bound for $\|v\|$, and there are only finitely many integral points in any bounded subset of $\mathcal{N}(X) \otimes \mathbb{R}$. \square

The following is the main result of this section.

Proposition 8.3. *The subset $\mathcal{P}_0(X) \subset \mathcal{N}(X) \otimes \mathbb{C}$ is open and the restriction*

$$\pi^{-1}(\mathcal{P}_0(X)) \xrightarrow{\pi} \mathcal{P}_0(X)$$

is a covering map.

Proof. Fix a norm $\|\cdot\|$ on the finite-dimensional complex vector space $\mathcal{N}(X) \otimes \mathbb{C}$ and take a point $\mathcal{U} \in \mathcal{P}_0(X)$. Lemma 8.1 shows that there is a constant $r > 0$ such that

$$|(u, v)| \leq r \|u\| |(\mathcal{U}, v)|$$

for all $u \in \mathcal{N}(X) \otimes \mathbb{C}$ and all $v \in \mathcal{N}(X) \otimes \mathbb{R}$ with $v^2 \geq 0$ and all $v \in \Delta(X)$. Given $\eta > 0$, define an open subset

$$B_\eta(\mathcal{U}) = \{\mathcal{U}' \in \mathcal{N}(X) \otimes \mathbb{C} : \|\mathcal{U}' - \mathcal{U}\| < \eta/r\} \subset \mathcal{N}(X) \otimes \mathbb{C}.$$

Then one has an implication

$$\mathcal{U}' \in B_\eta(\mathcal{U}) \implies |(\mathcal{U}', v) - (\mathcal{U}, v)| < \eta |(\mathcal{U}, v)|$$

for all $v \in \mathcal{N}(X) \otimes \mathbb{R}$ with $v^2 \geq 0$ and all $v \in \Delta(X)$. It follows from this that if $\eta < 1$ then any $\mathcal{U}' \in B_\eta(\mathcal{U})$ spans a positive definite two-plane in $\mathcal{N}(X) \otimes \mathbb{R}$ and satisfies $(\mathcal{U}', v) \neq 0$ for all $v \in \Delta(X)$. Hence $B_\eta(\mathcal{U}) \subset \mathcal{P}_0(X)$ and, in particular, $\mathcal{P}_0(X)$ is an open subset of $\mathcal{N}(X) \otimes \mathbb{C}$.

For each $\sigma \in \text{Stab}(X)$ with $\pi(\sigma) = \mathcal{U}$ define an open subset

$$C_\eta(\sigma) = \{\tau \in \pi^{-1}(B_\eta(\mathcal{U})) : f(\sigma, \tau) < \frac{1}{2}\} \subset \text{Stab}(X),$$

where f is the function introduced in Section 2. Lemma 5.1 shows that for any $\mathcal{U}' \in B_\eta(\mathcal{U})$ and any stable object $E \in \mathcal{D}(X)$

$$|(\mathcal{U}', v(E)) - (\mathcal{U}, v(E))| < \eta |(\mathcal{U}, v(E))|.$$

Thus by Theorem 2.4, for small enough $\eta > 0$ the map

$$\pi: C_\eta(\sigma) \longrightarrow B_\eta(\mathcal{U})$$

is onto, and hence, by Theorem 4.1 and Lemma 2.3, a homeomorphism. It follows from this that every stability condition in $\pi^{-1}(\mathcal{P}_0(X))$ is full (see Definition 4.2).

Fix a positive real number $\epsilon < \frac{1}{8}$ and assume that $\eta < \frac{1}{2} \sin(\pi\epsilon)$. Then, by Theorem 2.4 again, and using Lemma 4.5, for each $\sigma \in \pi^{-1}(\mathcal{U})$ the subset $C_\eta(\sigma)$ is

mapped homeomorphically by π onto $B_\eta(\mathcal{U})$. The final thing to check is that one has a disjoint union

$$\pi^{-1}(B_\eta(\mathcal{U})) = \bigcup_{\sigma \in \pi^{-1}(\mathcal{U})} C_\eta(\sigma).$$

The fact that the union is disjoint follows from Lemma 2.3. Suppose τ is a stability condition on $\mathcal{D}(X)$ with $\pi(\tau) = \mathcal{U}' \in B_\eta(\mathcal{U})$. If an object $E \in \mathcal{D}(X)$ is stable in τ then

$$|(\mathcal{U}', v(E)) - (\mathcal{U}, v(E))| < \eta |(\mathcal{U}, v(E))|.$$

This implies that

$$|(\mathcal{U}', v(E)) - (\mathcal{U}, v(E))| < \frac{\eta}{1-\eta} |(\mathcal{U}', v(E))| < 2\eta |(\mathcal{U}', v(E))|.$$

Applying Theorem 2.4 again gives a stability condition $\sigma \in \pi^{-1}(\mathcal{U})$ such that $f(\sigma, \tau) < \epsilon$, and then $\tau \in C_\eta(\sigma)$. \square

The following definition will be useful.

Definition 8.4. A connected component $\text{Stab}^*(X) \subset \text{Stab}(X)$ will be called good if it contains a point σ with $\pi(\sigma) \in \mathcal{P}_0(X)$. A stability condition $\sigma \in \text{Stab}(X)$ will be called good if it lies in a good component.

As observed in the proof of Proposition 8.3, a good component of $\text{Stab}(X)$ is also full. Indeed, if a connected component $\text{Stab}^*(X) \subset \text{Stab}(X)$ is good then by Proposition 8.3 the image of the map

$$\pi: \text{Stab}^*(X) \longrightarrow \mathcal{N}(X) \otimes \mathbb{C}$$

contains one of the two connected components of the open subset $\mathcal{P}_0(X)$. In particular, this image is not contained in a linear subspace.

The results of Sections 6 and 7 show that there do indeed exist good components of $\text{Stab}(X)$. Note also that if $\Phi \in \text{Aut } \mathcal{D}(X)$ is an autoequivalence then it acts on $\mathcal{N}(X)$ as an isometry, so if a stability condition $\sigma \in \text{Stab}(X)$ satisfies $\pi(\sigma) \in \mathcal{P}_0(X)$ then the same is true of $\Phi(\sigma)$. Thus $\text{Aut } \mathcal{D}(X)$ acts on the set of good stability conditions.

9. THE WALL AND CHAMBER STRUCTURE

Consider for a moment the problem of determining the set of μ_ω -stable sheaves on X as a function of the ample divisor class $\omega \in \text{Amp}(X)$. As is well-known, if one restricts attention to sheaves with fixed Mukai vector, the space $\text{Amp}(X)$ splits into a series of chambers in such a way that the set of μ_ω -stable sheaves with the given Mukai vector is constant in each chamber. The aim of this section is to prove a similar result for $\text{Stab}(X)$.

Suppose $\text{Stab}^*(X) \subset \text{Stab}(X)$ is a good component. The main result of this section is that if S is some finite set of objects of $\mathcal{D}(X)$, and $B \subset \text{Stab}^*(X)$ is a compact subset, then there is a finite collection of walls, which is to say codimension one submanifolds of B , such that as one varies the stability condition $\sigma \in B$, an element of S which is stable can only become unstable if one crosses a wall.

More generally it is not necessary to assume that the set $S \subset \mathcal{D}(X)$ is finite, only that its elements have bounded mass in the sense of the following definition.

Definition 9.1. A set of objects $S \subset \mathcal{D}(X)$ has bounded mass in a connected component $\text{Stab}^*(X) \subset \text{Stab}(X)$ if

$$\sup\{m_\sigma(E) : E \in S\} < \infty$$

for some point $\sigma \in \text{Stab}^*(X)$. Note that the fact that $\text{Stab}^*(X) \subset \text{Stab}(X)$ is connected implies that $d(\sigma, \tau) < \infty$ for all points $\sigma, \tau \in \text{Stab}^*(X)$, so that if this condition holds at some point $\sigma \in \text{Stab}^*(X)$ then it holds at all points.

The following easy result will be crucial.

Lemma 9.2. *Suppose the subset $S \subset \mathcal{D}(X)$ has bounded mass in a good component $\text{Stab}^*(X) \subset \text{Stab}(X)$. Then the set of Mukai vectors $\{v(E) : E \in S\}$ is finite.*

Proof. By assumption there is a $\sigma = (Z, \mathcal{P}) \in \text{Stab}^*(X)$ such that $\pi(\sigma) \in \mathcal{P}_0(X)$. Let $m > 0$ be such that $m_\sigma(E) < m$ for all $E \in S$. Then the stable factors A_1, \dots, A_n of an object $E \in S$ with respect to the stability condition σ must satisfy

$$\sum_i |Z(A_i)| < m.$$

By Lemma 5.1 and Lemma 8.2 there are only finitely many possibilities for the Mukai vectors $v(A_i)$, and hence only finitely many possibilities for the Mukai vector $v(E)$. \square

The next result gives the claimed wall and chamber structure.

Proposition 9.3. *Suppose the subset $S \subset \mathcal{D}(X)$ has bounded mass in a good component $\text{Stab}^*(X) \subset \text{Stab}(X)$, and fix a compact subset $B \subset \text{Stab}^*(X)$. Then there is a finite collection $\{W_\gamma : \gamma \in \Gamma\}$ of (not necessarily closed) real codimension one submanifolds of $\text{Stab}^*(X)$ such that any connected component*

$$C \subset B \setminus \bigcup_{\gamma \in \Gamma} W_\gamma$$

has the following property: if $E \in S$ is semistable in σ for some $\sigma \in C$, then E is semistable in σ for all $\sigma \in C$; if moreover $E \in S$ has primitive Mukai vector then E is stable in σ for all $\sigma \in C$.

Proof. Let $T \subset \mathcal{D}(X)$ be the set of nonzero objects $A \in \mathcal{D}(X)$ such that for some $\sigma \in B$ and some $E \in S$ one has $m_\sigma(A) \leq m_\sigma(E)$. The fact that B is compact implies that the quotient $m_\tau(E)/m_\sigma(E)$ is uniformly bounded for all nonzero $E \in \mathcal{D}(X)$, and for all $\sigma, \tau \in B$, so the subset $T \subset \mathcal{D}(X)$ has bounded mass in $\text{Stab}^*(X)$. Note that if A is a semistable factor of an object $E \in S$ in some stability condition $\sigma \in B$ then A is an element of T .

Let $\{v_i : i \in I\}$ be the finite set of Mukai vectors of objects of T and let Γ be the set of pairs $i, j \in I$ such that v_i and v_j do not lie on the same real line in $\mathcal{N}(X) \otimes \mathbb{R}$. For each $\gamma = (i, j) \in \Gamma$ define

$$W_\gamma = \{\sigma = (Z, \mathcal{P}) \in \text{Stab}^*(X) : Z(v_i)/Z(v_j) \in \mathbb{R}_{>0}\}.$$

Since $\text{Stab}^*(X)$ is a good component, and hence full, the map $\pi: \text{Stab}^*(X) \rightarrow \mathcal{N}(X) \otimes \mathbb{C}$ is a local homeomorphism. Since W_γ is the inverse image under π of an open subset of a real quadric in $\mathcal{N}(X) \otimes \mathbb{C}$, it follows that each W_γ is a real codimension one submanifold of $\text{Stab}^*(X)$.

Suppose that $C \subset B$ is a connected component of

$$B \setminus \bigcup_{\gamma \in \Gamma} W_\gamma$$

and fix an object $E \in S$. Consider the subset $V \subset C$ consisting of points at which E is semistable, and assume that V is nonempty. The definition of the topology on $\text{Stab}(X)$ ensures that V is a closed subset of C . The next step is to show that V is also open in C and hence that $V = C$.

Suppose then that $\sigma = (Z, \mathcal{P}) \in V$ is such that $E \in \mathcal{P}(\phi)$ for some $\phi \in \mathbb{R}$. Take $0 < \eta < \frac{1}{8}$ such that the open neighbourhood

$$U = \{\tau \in \text{Stab}(X) : d(\sigma, \tau) < \eta\}$$

is contained in C . It will be enough to show that $U \subset V$.

Note that the assumption $\eta < \frac{1}{8}$ ensures that if A is a semistable factor of E in some stability condition in U then A lies in the abelian subcategory

$$\mathcal{A} = \mathcal{P}((\phi - \frac{1}{2}, \phi + \frac{1}{2}]) \subset \mathcal{D}(X),$$

and that moreover, for any other stability condition $\tau = (W, \mathcal{Q}) \in U$ the central charge $W(A)$ lies in the half-plane

$$H_\phi = \{r \exp(i\pi\psi) : r > 0 \text{ and } \phi - \frac{1}{2} < \psi < \phi + \frac{1}{2}\}.$$

Suppose for a contradiction that E is unstable at some point $\sigma' = (Z', \mathcal{P}') \in U$. Then there is a semistable factor A of E in the stability condition σ' which is a subobject of E in the category \mathcal{A} , and which satisfies $\text{Im } Z'(A)/Z'(E) > 0$. As

$\tau = (W, \mathcal{Q})$ varies in U the complex numbers $W(A)$ and $W(E)$ remain in H_ϕ , and therefore, since U is contained in a connected component of

$$B \setminus \bigcup_{\gamma \in \Gamma} W_\gamma$$

and A and E are objects of T , it follows that $\text{Im } W(A)/W(E) > 0$ for all $\tau = (W, \mathcal{Q}) \in U$. This contradicts the fact that E is semistable at σ thus proving the claim.

To complete the proof of the Proposition, suppose an object $E \in S$ has primitive Mukai vector $v(E) \in \mathcal{N}(X)$, and suppose E is semistable but not stable at some point $\sigma \in B$. Each stable factor A of E in the stability condition σ has mass less than E so has Mukai vector v_i for some $i \in I$. Since $v(E)$ is primitive, not all the Mukai vectors of the stable factors of E are multiples of the Mukai vector of E . But the phases of all the stable factors are the same so one must have $\sigma \in W_\gamma$ for some pair $\gamma = (i, j) \in \Gamma$. \square

Proposition 9.4. *Suppose the subset $S \subset \mathcal{D}(X)$ has bounded mass in a good component $\text{Stab}^*(X) \subset \text{Stab}(X)$ and that each object $E \in S$ has primitive Mukai vector $v(E) \in \mathcal{N}(X)$. Then the set of points $\sigma \in \text{Stab}^*(X)$ at which all objects of S are stable is open in $\text{Stab}^*(X)$.*

Proof. It will be enough to fix a compact subset $B \subset \text{Stab}^*(X)$ and prove that the set $F = \{\sigma \in B : \text{not every } E \in S \text{ is stable in } \sigma\}$ is a closed subset of B . Let T be the set of all semistable factors of objects of S in all stability conditions of B . It was observed in the proof of Proposition 9.3 that this set has bounded mass in $\text{Stab}^*(X)$. Consider the corresponding chamber decomposition given by Proposition 9.3. The aim is to show that F is closed by proving that it is the union of the closures of those chambers C in which some object $E \in S$ is not stable.

Take a stability condition $\sigma = (Z, \mathcal{P}) \in F$ lying in the closure of a finite set of chambers $C_j \subset B$. Take an object $E \in S$ such that E is not stable in σ . If E is not semistable in σ then E is not semistable in an open neighbourhood of σ , so E is not stable in each chamber C_j . The other possibility is that E is semistable of some phase ϕ in σ but not stable. Then there is a short exact sequence

$$0 \longrightarrow A \longrightarrow E \longrightarrow B \longrightarrow 0$$

in the abelian subcategory $\mathcal{P}(\phi) \subset \mathcal{D}(X)$. Since $v(E)$ is primitive, the Mukai vectors of A and B are not multiples of each other, and since π is a local homeomorphism it follows that there are points $\tau = (W, \mathcal{Q})$ arbitrarily close to σ for which $\text{Im } W(A)/W(B) > 0$. This implies that E must be unstable in one of the chambers C_j . Thus either way σ lies in the closure of a chamber $C \subset B$ in which E is unstable.

For the converse suppose that some object $E \in S$ is unstable in a chamber $C \subset B$. Take a stability condition $\sigma \in C$ and let the semistable factors of E in σ be A_1, \dots, A_n with phases $\phi(A_1) > \dots > \phi(A_n)$. Then each A_i is an object of T , so by the construction of Proposition 9.3, the objects A_i are semistable at each point of the closure of C and satisfy $\phi(A_1) \geq \dots \geq \phi(A_n)$. It follows that E is not stable at any point of the closure of C , which is to say that F contains the closure of C . \square

10. CLASSIFYING STABILITY CONDITIONS

It follows from Lemma 6.3 that all the stability conditions constructed in Section 6 have the property that for each point $x \in X$, the corresponding skyscraper sheaf \mathcal{O}_x is stable in σ of phase one. The next step is to prove a converse to this result.

Lemma 10.1. *Suppose $\sigma = (Z, \mathcal{P}) \in \text{Stab}(X)$ is a stability condition on X such that for each point $x \in X$ the sheaf \mathcal{O}_x is stable in σ of phase one. Let E be an object of $\mathcal{D}(X)$. Then*

- (a) *if $E \in \mathcal{P}((0, 1])$ then the cohomology sheaves $H^i(E)$ vanish unless $i \in \{-1, 0\}$, and moreover the sheaf $H^{-1}(E)$ is torsion-free,*
- (b) *if $E \in \mathcal{P}(1)$ is stable then either $E = \mathcal{O}_x$ for some $x \in X$, or $E[-1]$ is a locally-free sheaf,*
- (c) *if $E \in \text{Coh}(X)$ is a sheaf then $E \in \mathcal{P}((-1, 1])$; if E is a torsion sheaf then $E \in \mathcal{P}((0, 1])$,*
- (d) *the pair of subcategories*

$$\mathcal{T} = \text{Coh}(X) \cap \mathcal{P}((0, 1]) \text{ and } \mathcal{F} = \text{Coh}(X) \cap \mathcal{P}((-1, 0])$$

defines a torsion pair on the category of coherent sheaves $\text{Coh}(X)$ and moreover the category $\mathcal{P}((0, 1])$ is the corresponding tilt.

Proof. Suppose that E is stable of phase ϕ for some $0 < \phi < 1$. For any $x \in X$ the sheaf \mathcal{O}_x is stable of phase 1, so that $\text{Hom}_X^i(E, \mathcal{O}_x) = 0$ for $i < 0$, and Serre duality gives

$$\text{Hom}_X^i(E, \mathcal{O}_x) = \text{Hom}_X^{2-i}(\mathcal{O}_x, E)^* = 0$$

for $i \geq 2$. Taking a locally-free resolution of E and truncating (see for example [4, Proposition 5.4]) it follows that E is isomorphic to a length two complex of locally-free sheaves, so that E satisfies the conclusions of part (a).

If E is stable of phase $\phi = 1$ and E is not a skyscraper sheaf, then there cannot be any nonzero maps $E \rightarrow \mathcal{O}_x$ or $\mathcal{O}_x \rightarrow E$, so the same argument shows that $\text{Hom}_X^i(E, \mathcal{O}_x) = 0$ unless $i = 1$, and hence $E[-1]$ is locally-free. Any object of $\mathcal{P}((0, 1])$ has a filtration by stable objects with phases in the interval $(0, 1]$ so this proves (a) and (b).

Suppose now that E is a sheaf on X . For any object $A \in \mathcal{P}(> 1)$ part (a) shows that $H^i(A) = 0$ for $i \geq 0$ so that $\text{Hom}_X(A, E) = 0$. Similarly, if $B \in \mathcal{P}(\leq -1)$ then $H^i(B) = 0$ for $i \leq 0$ so that $\text{Hom}_X(E, B) = 0$. It follows that $E \in \mathcal{P}((-1, 1])$ which gives the first half of (c).

Given a sheaf E on X , by the first part of (c) there is a triangle

$$\begin{array}{ccc} D & \xrightarrow{\quad} & E \\ & \swarrow \text{---} & \searrow \\ & F & \end{array}$$

with $D \in \mathcal{P}((0, 1])$ and $F \in \mathcal{P}((-1, 0])$. By part (a) one has $H^i(D) = 0$ unless $i \in \{-1, 0\}$ and $H^i(F) = 0$ unless $i \in \{0, 1\}$. Taking the long exact sequence in cohomology one concludes that D and F must both be sheaves. This shows that \mathcal{T}, \mathcal{F} is a torsion pair and it is immediate that $\mathcal{P}((0, 1])$ is the corresponding tilt. Moreover, by part (a) again the sheaf F is torsion-free so that if E is torsion then $E \in \mathcal{P}((0, 1])$, which completes the proof of (c). \square

Certain technical problems arise if one tries to classify all stability conditions satisfying the conditions of Lemma 10.1. However the extra assumption that σ is good (Definition 8.4) leads to a complete classification.

Definition 10.2. Let $U(X) \subset \text{Stab}(X)$ be the subset consisting of good stability conditions $\sigma \in \text{Stab}(X)$ such that for each point $x \in X$ the sheaf \mathcal{O}_x is stable in σ of the same phase.

It follows from Proposition 9.4 that $U(X) \subset \text{Stab}(X)$ is open.

Recall from the introduction that $\text{GL}^+(2, \mathbb{R})$ acts freely on $\mathcal{P}(X)$ by change of basis in the positive two plane spanned by the real and imaginary parts of a vector $\bar{U} \in \mathcal{N}(X) \otimes \mathbb{C}$. A section of this action is provided by the submanifold

$$\mathcal{Q}(X) = \{ \bar{U} \in \mathcal{P}(X) : (\bar{U}, \bar{U}) = 0, (\bar{U}, \bar{U}) > 0, r(\bar{U}) = 1 \} \subset \mathcal{N}(X) \otimes \mathbb{C},$$

where $r(\bar{U})$ is the projection of $\bar{U} \in \mathcal{N}(X) \otimes \mathbb{C} \subset H^*(X, \mathbb{C})$ onto $H^0(X, \mathbb{C})$. Note that $\mathcal{Q}(X)$ can be identified with the tube domain

$$\{ \beta + i\omega \in \text{NS}(X) \otimes \mathbb{C} : \omega^2 > 0 \}$$

via the exponential map $\bar{U} = \exp(\beta + i\omega)$.

Before stating the next result, let us agree to say that a stability condition $\sigma = (Z, \mathcal{P})$ on $\mathcal{D}(X)$ arises from the construction of Section 6 if there is a pair $\beta, \omega \in \text{NS}(X) \otimes \mathbb{R}$ with $\omega \in \text{Amp}(X)$ such that the central charge Z is given by

$$Z(E) = (\exp(\beta + i\omega), v(E)),$$

and that moreover the heart $\mathcal{P}((0, 1])$ of σ is the abelian subcategory $\mathcal{A}(\beta, \omega) \subset \mathcal{D}(X)$ defined in Section 6. Note that in this case the pair β, ω is uniquely defined by σ , and that conversely, by Proposition 3.5, given any such pair there is at most one stability condition arising from β, ω via the construction of Section 6.

Proposition 10.3. *If $\sigma = (Z, \mathcal{P}) \in U(X)$ then there is a unique element $g \in \tilde{\text{GL}}^+(2, \mathbb{R})$, such that σg arises from the construction of Section 6.*

Proof. The proof is best broken into a sequence of steps.

STEP 1. Applying an element of $\tilde{\text{GL}}^+(2, \mathbb{R})$ one can assume that $\mathcal{O}_x \in \mathcal{P}(1)$ for all $x \in X$. Since $Z(\mathcal{O}_x)$ lies on the real axis, the central charge Z of an object $E \in \mathcal{D}(X)$ with Mukai vector (r, Δ, s) must satisfy

$$\text{Im } Z(E) = \Delta \cdot \omega - rt,$$

for some $\omega \in \text{NS}(X) \otimes \mathbb{R}$ and some $t \in \mathbb{R}$. The first step is to show that $\omega \in \text{Amp}(X)$ is ample.

Suppose $C \subset X$ is a curve. Lemma 10.1(c) shows that the torsion sheaf \mathcal{O}_C lies in the subcategory $\mathcal{P}((0, 1])$. If $Z(\mathcal{O}_C)$ lies on the real axis it follows that $\mathcal{O}_C \in \mathcal{P}(1)$ which is impossible by Lemma 10.1(b). Thus $\text{Im } Z(\mathcal{O}_C) = C \cdot \omega > 0$. Now σ is a good stability condition, and hence full, so the map

$$\pi: \text{Stab}(X) \rightarrow \mathcal{N}(X) \otimes \mathbb{C}$$

is a local homeomorphism near σ . Since $U(X)$ is open one may deform ω a little and still conclude that $C \cdot \omega > 0$ for all curves $C \subset X$. Thus ω lies in the interior of the nef cone in $H^2(X, \mathbb{R})$ and hence is ample.

STEP 2. Consider the torsion pair $(\mathcal{T}, \mathcal{F})$ of Lemma 10.1(d). Note that by Lemma 10.1(c) all torsion sheaves lie in \mathcal{T} . The next claim is that if E is a μ_ω -stable torsion-free sheaf then either $E \in \mathcal{T}$ or $E \in \mathcal{F}$ depending on whether $\text{Im } Z(E) > 0$ or $\text{Im } Z(E) \leq 0$.

Given such an E there is a short exact sequence of sheaves

$$0 \longrightarrow D \longrightarrow E \longrightarrow F \longrightarrow 0$$

with $D \in \mathcal{T}$ and $F \in \mathcal{F}$. Assume D and F are both nonzero. By Lemma 10.1(c), the sheaf F must be torsion-free. Now one has $\mu_\omega(D) \geq t$ and $\mu_\omega(F) \leq t$ and since E is μ_ω -stable, this gives a contradiction. Thus E lies in either \mathcal{T} or \mathcal{F} . Clearly, if $\text{Im } Z(E) > 0$ then $E \in \mathcal{T}$ and if $\text{Im } Z(E) < 0$ then $E \in \mathcal{F}$.

Suppose that $\text{Im } Z(E) = 0$. It must be shown that $E \in \mathcal{F}$. Suppose for a contradiction that $E \in \mathcal{T}$. Since $Z(E) \in \mathbb{R}$ it follows that $E \in \mathcal{P}(1)$. Consider a nonzero map $f: E \rightarrow \mathcal{O}_x$ and let F be its kernel in the category $\text{Coh}(X)$. Since \mathcal{O}_x is stable, the map f is a surjection in the abelian category $\mathcal{P}(1)$, so the object

F also lies in $\mathcal{P}(1)$ and hence in \mathcal{T} . Repeating must eventually give $Z(F) \in \mathbb{R}_{>0}$ which contradicts $F \in \mathcal{P}((0, 1])$.

STEP 3. The next step is to show that $\bar{U} = \pi(\sigma) \in \mathcal{P}(X)$, which is to say that the real and imaginary parts of \bar{U} span a positive definite two-plane in $\mathcal{N}(X) \otimes \mathbb{R}$. Equivalently one must show that $\text{Ker}(Z) \subset \mathcal{N}(X) \otimes \mathbb{R}$ is negative definite.

Suppose for a contradiction that there is a $v \in \text{NS}(X) \otimes \mathbb{R}$ such that $Z(v) = 0$ and $(v, v) \geq 0$. Since $U(X)$ is open one can deform σ and assume that $(v, v) > 0$ and $v \in \text{NS}(X) \otimes \mathbb{Q}$, and hence by removing denominators, that $v \in \text{NS}(X)$.

If $r(v) = 0$ then the assumption $(v, v) > 0$ implies that either v or $-v$ is the Mukai vector of a torsion sheaf T on X . But by Lemma 10.1(c) one has $T \in \mathcal{P}((0, 1])$ so that $Z(v) \neq 0$. Otherwise one can assume that $r(v) > 0$. But Theorem 5.3 then shows that v is the Mukai vector of a μ_ω -semistable torsion-free sheaf E . By Step 2 it follows that either $E \in \mathcal{T}$ or $E \in \mathcal{F}$ and hence $Z(v) \neq 0$.

STEP 4. Now one may apply an element of $g \in \tilde{\text{GL}}^+(2, \mathbb{R})$ and assume that $\bar{U} \in \mathcal{Q}(X)$ so that

$$Z(E) = (\exp(\beta + i\omega), v(E)),$$

for some pair $\beta, \omega \in \text{NS}(X) \otimes \mathbb{R}$ with $\omega^2 > 0$. The group $\text{GL}^+(2, \mathbb{R})$ acts freely on $\mathcal{P}(X)$ and the submanifold $\mathcal{Q}(X)$ is a section of this action, so the element g is defined uniquely up to the kernel of the homomorphism $\tilde{\text{GL}}^+(2, \mathbb{R}) \rightarrow \text{GL}^+(2, \mathbb{R})$. This kernel acts on $\text{Stab}(X)$ by even shifts, so if one also assumes that each object \mathcal{O}_x is stable in g of phase 1 then g is uniquely determined.

By Step 1 one knows that ω is ample. Then Step 2 shows that the torsion pair $(\mathcal{T}, \mathcal{F})$ of Lemma 10.1(d) coincides with the torsion pair of Lemma 6.1. It follows immediately that $\mathcal{P}((0, 1]) = \mathcal{A}(\beta, \omega)$ and hence that σ arises from the construction of Section 6. \square

11. THE OPEN SUBSET $U(X)$

Following Proposition 10.3 it is now possible to give a precise description of the set $U(X) \subset \text{Stab}(X)$ defined in the last section. Firstly one knows that the action of $\tilde{\text{GL}}^+(2, \mathbb{R})$ on $U(X)$ is free. A section of this action is defined by the submanifold

$$V(X) = \{\sigma \in U(X) : \pi(\sigma) \in \mathcal{Q}(X) \text{ and each } \mathcal{O}_x \text{ is stable in } \sigma \text{ of phase } 1\}.$$

The proof of Proposition 10.3 showed that $V(X)$ is precisely the set of stability conditions arising from the construction of Section 6. In particular, the map π is injective when restricted to $V(X)$.

To understand the image of $V(X)$ recall first that $\mathcal{Q}(X)$ can be identified with the tube domain

$$\{\beta + i\omega \in \text{NS}(X) \otimes \mathbb{C} : \omega^2 > 0\}$$

via the exponential map $\mathcal{U} = \exp(\beta + i\omega)$. The image of the complexified ample cone under this map defines an open subset

$$\mathcal{K}(X) = \{\mathcal{U} = \exp(\beta + i\omega) \in \mathcal{Q}(X) : \omega \in \text{Amp}(X)\} \subset \mathcal{Q}(X).$$

Define

$$\Delta^+(X) = \{\delta \in \mathcal{N}(X) : (\delta, \delta) = -2 \text{ and } r(\delta) > 0\} \subset \Delta(X).$$

Proposition 11.2 below shows that the image of $V(X)$ under π is the subset

$$\mathcal{L}(X) = \{\mathcal{U} \in \mathcal{K}(X) : (\mathcal{U}, \delta) \notin \mathbb{R}_{\leq 0} \text{ for all } \delta \in \Delta^+(X)\} \subset \mathcal{Q}(X).$$

Note that this is in fact a subset of $\mathcal{P}_0(X)$ since if $\delta \in \Delta(X)$ satisfies $r(\delta) = 0$ then $\delta = \pm(0, C, n)$ for some (-2) -curve $C \subset X$ and it follows that $\text{Im}(\mathcal{U}, \delta) \neq 0$ for any $\mathcal{U} \in \mathcal{K}(X)$.

Lemma 11.1. *The subset $\mathcal{L}(X) \subset \mathcal{Q}(X)$ is open and contractible.*

Proof. To show that $\mathcal{L}(X)$ is open note that it is the complement in $\mathcal{K}(X)$ of the real halfplanes

$$H(\delta) = \{\mathcal{U} \in \mathcal{K}(X) : (\mathcal{U}, \delta) \in \mathbb{R}_{\leq 0}\}$$

for elements $\delta \in \Delta^+(X)$. Thus it will be enough to show that these hyperplanes are locally finite in $\mathcal{K}(X)$.

Fix a bounded region $B \subset \mathcal{K}(X)$ and take an element $\mathcal{U} = \exp(\beta + i\omega) \in B$. Suppose that $\mathcal{U} \in H(\delta)$ for some $\delta = (r, \Delta, s) \in \Delta^+(X)$. According to the formula (\star) of Section 6

$$\frac{1}{2r} \left((\Delta^2 - 2rs) + r^2\omega^2 - (\Delta - r\beta)^2 \right) + i(\Delta - r\beta) \cdot \omega = (\mathcal{U}, \delta) \in \mathbb{R}_{\leq 0}$$

Thus $\Delta - r\beta$ is an element of the sublattice $\omega^\perp \subset \text{NS}(X) \otimes \mathbb{R}$, which is negative definite. Now $\Delta^2 - 2rs = -2$, so that

$$r^2\omega^2 - (\Delta - r\beta)^2 \leq 2,$$

and since ω is constrained to lie in a bounded region of $\text{NS}(X) \otimes \mathbb{R}$, it follows that there are only finitely many possible choices for r . Also $\Delta - r\beta$ lies in a bounded region of $\omega^\perp \subset \text{NS}(X) \otimes \mathbb{R}$, and since β is constrained to lie in a bounded region of $\text{NS}(X) \otimes \mathbb{R}$ it follows that there are only finitely many possibilities for Δ , and hence only finitely many hyperplanes $H(\delta)$ which meet the region B .

To see that $\mathcal{L}(X)$ is contractible note that if $\exp(\beta + i\omega)$ is an element of $\mathcal{L}(X)$ then so is $\exp(\beta + it\omega)$ for every $t > 1$. Thus $\mathcal{L}(X)$ has a deformation retraction onto the subset

$$\{\exp(\beta + i\omega) \in \mathcal{K}(X) : \omega^2 > 2\} \subset \mathcal{L}(X)$$

and hence is contractible. \square

The following result is a stronger version of Proposition 10.3.

Proposition 11.2. *The map π restricts to give a homeomorphism*

$$\pi: V(X) \longrightarrow \mathcal{L}(X).$$

The stability condition in $V(X)$ corresponding to a point $\exp(\beta + i\omega) \in \mathcal{L}(X)$ is given by the construction of Section 6.

Proof. As noted above, any stability condition $\sigma \in V(X)$ arises from the construction of Section 6, so the map π is injective when restricted to $V(X) \subset \text{Stab}(X)$ and hence gives a homeomorphism $V(X) \rightarrow \pi(V(X))$. Moreover, for any stability condition $\sigma \in V(X)$ one has $\pi(\sigma) = \exp(\beta + i\omega) \in \mathcal{K}(X)$. Theorem 5.3 ensures that for any $\delta \in \Delta^+(X)$ there is a μ_ω -semistable torsion-free sheaf E with $v(E) = \delta$. Since σ arises from the construction of Section 6 one either has $E \in \mathcal{P}((0, 1])$ or $E[1] \in \mathcal{P}((0, 1])$. Either way, $Z(E) \neq 0$. This shows that $\pi(\sigma) \in \mathcal{L}(X)$. Thus π maps $V(X)$ into $\mathcal{L}(X)$, and it only remains to show that this map is onto. But Proposition 9.4 shows that $\pi(V(X))$ is an open subset of $\mathcal{L}(X)$, so since $\mathcal{L}(X)$ is connected it will be enough to show that it is also closed.

Suppose $\mathcal{U} = \exp(\beta + i\omega) \in \mathcal{L}(X)$ lies in the closure of $\pi(V(X))$. Since π is a covering over $\mathcal{L}(X)$ there is a stability condition σ lying in the closure of $V(X)$ with $\pi(\sigma) = \mathcal{U}$. If σ is not an element of $V(X)$ there must be some skyscraper sheaf $\mathcal{O}_x \in \mathcal{P}(1)$ which is semistable in σ but not stable. Since ω is ample, any class $v \in \mathcal{N}(X)$ such that $\text{Im } Z(v) = 0$ must either be a multiple of $v(\mathcal{O}_x)$ or have nonzero rank. Thus there must be a stable factor A of \mathcal{O}_x with $r(A) > 0$. Then $Z(A)$ lies on the negative real axis. The argument of Lemma 6.2 shows that A must be spherical, so that $v(A) \in \Delta^+(X)$, and since $\mathcal{U} \in \mathcal{L}(X)$ this is impossible. Thus $\sigma \in V(X)$ and $\pi(V(X))$ is closed in $\mathcal{L}(X)$. \square

Recall that $\mathcal{P}^+(X) \subset \mathcal{P}(X)$ is the connected component containing $\mathcal{K}(X)$, and that $\mathcal{P}_0^+(X) = \mathcal{P}^+(X) \cap \mathcal{P}_0(X)$. It is worth recording here the following easy consequence of what has been proved so far.

Corollary 11.3. *If $\sigma \in U(X)$ then $\pi(\sigma) \in \mathcal{P}_0^+(X)$. Moreover σ is determined by $\pi(\sigma)$ up to even shifts.*

Proof. As shown in Proposition 10.3, for $\sigma \in U(X)$ there is a unique $g \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ such that $\sigma g \in V(X)$. The first statement then follows from Proposition 11.2 since $\mathcal{P}_0^+(X)$ is invariant under $\mathrm{GL}^+(2, \mathbb{R})$. For the second statement note that the only elements of the group $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ which fix the central charge of a stability condition are double shifts. \square

Note that Proposition 11.2 shows that the stability function of Lemma 6.2 always has the Harder-Narasimhan property. Another consequence of Proposition 11.2 is that the set $V(X)$, and hence also $U(X)$, is connected. Thus all the stability conditions constructed in Section 6 are contained in the same connected component of $\mathrm{Stab}(X)$.

Definition 11.4. Let $\mathrm{Stab}^\dagger(X) \subset \mathrm{Stab}(X)$ be the unique connected component containing the set $U(X)$.

The results of this section give a precise description of the stability conditions lying in the open subset $U(X) \subset \mathrm{Stab}(X)$. The next step is to analyse the boundary of this set.

12. THE BOUNDARY OF $U(X)$

The aim of this section is to study the boundary $\partial U(X) = \overline{U(X)} \setminus U(X)$ of the open subset $U(X) \subset \mathrm{Stab}(X)$ introduced in the last section. The results of Section 9 show that $\partial U(X)$ is contained in a locally-finite union of codimension one, real submanifolds of $\mathrm{Stab}(X)$. A point of the boundary will be called general if it lies on only one of these submanifolds.

Theorem 12.1. *Suppose $\sigma = (Z, \mathcal{P}) \in \partial U(X)$ is a general point of the boundary of $U(X)$. Then exactly one of the following possibilities holds.*

(A^+) *There is a rank r spherical vector bundle A such that the only stable factors of the objects $\{\mathcal{O}_x : x \in X\}$ in the stability condition σ are A and $T_A(\mathcal{O}_x)$. Thus the Jordan-Hölder filtration of each \mathcal{O}_x is given by*

$$0 \longrightarrow A^{\oplus r} \longrightarrow \mathcal{O}_x \longrightarrow T_A(\mathcal{O}_x) \longrightarrow 0.$$

(A^-) *There is a rank r spherical vector bundle A such that the only stable factors of the objects $\{\mathcal{O}_x : x \in X\}$ in the stability condition σ are $A[2]$ and $T_A^{-1}(\mathcal{O}_x)$. Thus the Jordan-Hölder filtration of each \mathcal{O}_x is given by*

$$0 \longrightarrow T_A^{-1}(\mathcal{O}_x) \longrightarrow \mathcal{O}_x \longrightarrow A^{\oplus r}[2] \longrightarrow 0.$$

(C_k) *There is a non-singular rational curve $C \subset X$ and an integer k such that \mathcal{O}_x is stable in the stability condition σ for $x \notin C$ and such that the Jordan-Hölder filtration of \mathcal{O}_x for $x \in C$ is*

$$0 \longrightarrow \mathcal{O}_C(k+1) \longrightarrow \mathcal{O}_x \longrightarrow \mathcal{O}_C(k)[1] \longrightarrow 0.$$

Moreover, a stability condition σ satisfies A^- precisely if the stability condition $T_A^2(\sigma)$ satisfies A^+ , and similarly a stability condition σ satisfies C_k precisely if the stability condition $T_{\mathcal{O}_C(k)}(\sigma)$ satisfies C_{k-1} .

Proof. If $\sigma \in \partial U(X)$ then each skyscraper sheaf \mathcal{O}_x is semistable of the same phase. Applying an element of $\tilde{\text{GL}}^+(2, \mathbb{R})$ it is enough to consider the case when this phase is 1. Write $W(X)$ for the subset of $U(X)$ consisting of stability conditions for which each object \mathcal{O}_x is stable of phase 1. Since $U(X)$ is invariant under the action of $\tilde{\text{GL}}^+(2, \mathbb{R})$ it follows that σ lies in the closure of $W(X)$.

STEP 1. Since $\sigma \notin U(X)$ there must be some $x \in X$ such that \mathcal{O}_x is semistable but not stable. Applying Lemma 12.2 below shows that there is a stable, spherical object $A \in \mathcal{P}(1)$ with a nonzero map either $A \rightarrow \mathcal{O}_x$ or $\mathcal{O}_x \rightarrow A$.

Since σ is general, the boundary of $U(X)$ at σ is a codimension one submanifold of $\text{Stab}(X)$ at σ , which is given by $Z(A)/Z(\mathcal{O}_x) \in \mathbb{R}_{>0}$. Thus if E is any object of $\mathcal{D}(X)$ one can move a little bit along the boundary of $U(X)$ and assume that $Z(E) \notin \mathbb{R}$, unless of course the Mukai vector $v(E)$ is a linear combination (over \mathbb{R}) of the vectors $v(A)$ and $v(\mathcal{O}_x)$.

Let $L \subset \mathcal{N}(X)$ be the rank 2 sublattice spanned by $v(A)$ and $v(\mathcal{O}_x)$. The fact that A is spherical implies that this lattice is primitive: if $v \in \mathcal{N}(X)$ lies in the linear span of the vectors $v(A)$ and $v(\mathcal{O}_x)$ over \mathbb{R} then it also lies in the integral span, and hence is an element of L . Indeed one can write $v(A) = (r, \Delta, s)$ and $v(\mathcal{O}_x) = (0, 0, 1)$. If $v = \lambda v(A) + \mu v(\mathcal{O}_x)$ lies in $\mathcal{N}(X)$ then $\lambda r \in \mathbb{Z}$, $\lambda \Delta \in \text{NS}(X)$ and $\lambda s - \mu \in \mathbb{Z}$. Clearly λ and μ are rational. Since $\Delta^2 - 2rs = -2$ the elements $\Delta \in \text{NS}(X)$ and $r \in \mathbb{Z}$ have no common divisor, so it follows that λ and hence μ are integral.

STEP 2. Assume first that there is a nonzero map $A \rightarrow \mathcal{O}_x$. Since σ lies in the closure of $W(X)$, and $v(A)$ is primitive, by Proposition 9.4 there are points $\tau = (W, \mathcal{Q}) \in W(X)$ arbitrarily close to σ such that the object A is also stable in τ . Thus one can take $f(\sigma, \tau) < \epsilon$ for some small $\epsilon > 0$. The nonzero map $A \rightarrow \mathcal{O}_x$ then implies that $A \in \mathcal{Q}(\phi)$ with $1 - \epsilon < \phi < 1$ and it follows from Lemma 10.1(a)

that A fits into a triangle

$$\begin{array}{ccc} D & \longrightarrow & A \\ & \swarrow \text{dashed} & \searrow \\ & E & \end{array}$$

with $D = H^{-1}(A)[1]$ and $E = H^0(A)$.

Suppose for a contradiction that D and E are both nonzero. Lemma 10.1(c) implies that $E \in \mathcal{Q}((-1, 1])$ for all stability conditions $(W, \mathcal{Q}) \in W(X)$, and hence $E \in \mathcal{P}([-1, -1])$. Similarly $D \in \mathcal{P}([0, 2])$. Lemma 12.3 below shows that $D \in \mathcal{P}(0)$ and $E \in \mathcal{P}(1)$. But now one could replace σ by another nearby point of the boundary of $U(X)$ and repeat the argument. Since $Z(D)$ and $Z(E)$ always have to lie on the real axis, and σ was assumed to be general, it follows that the Mukai vectors of D and E must lie in the rank two sublattice L , so we can write

$$v(D) = \lambda_D v(A) + \mu_D v(\mathcal{O}_x), \quad v(E) = \lambda_E v(A) + \mu_E v(\mathcal{O}_x)$$

for integers $\lambda_D, \lambda_E, \mu_D, \mu_E \in \mathbb{Z}$, with $\lambda_D + \lambda_E = 1$ and $\mu_D + \mu_E = 0$.

Moving again to the nearby stability condition $\tau = (W, \mathcal{Q}) \in W(X)$ considered above, Lemma 10.1(c) shows that $D \in \mathcal{Q}((0, \epsilon))$ and $E \in \mathcal{Q}((1 - \epsilon, 1])$. Also $A \in \mathcal{Q}(\phi)$ with $1 - \epsilon < \phi < 1$ and of course $\mathcal{O}_x \in \mathcal{Q}(1)$. Thus the imaginary part of W is positive on D and A , non-negative on E and zero on \mathcal{O}_x . It follows that $\lambda_D > 0$ and $\lambda_E \geq 0$. By Lemma 12.4 the sheaf E is rigid and so $v(E)^2 < 0$. Since $v(\mathcal{O}_x)^2 = 0$ it follows that $\lambda_E > 0$ also which gives a contradiction. Thus one of D or E is zero.

STEP 3. If $E = 0$ then $A = H^{-1}(A)[1]$ is concentrated in degree -1 and so there could not be a nonzero map $A \rightarrow \mathcal{O}_x$. Thus $D = 0$ and hence A is a sheaf. If A has positive rank then there is a short exact sequence of sheaves

$$0 \longrightarrow T \longrightarrow A \longrightarrow Q \longrightarrow 0.$$

with T torsion and Q torsion-free. Since σ lies in the closure of $W(X)$, Lemma 10.1(c) shows that $T \in \mathcal{P}(\geq 0)$ and $Q \in \mathcal{P}([-1, 1])$. Lemma 12.3 then shows that $T \in \mathcal{P}(0)$. Again, since σ is general it follows that $v(T)$ lies in the sublattice L . But $r(A) > 0$ so this means that $v(T)$ is a multiple of $v(\mathcal{O}_x)$ and so T is supported in dimension 0. This too is impossible because all zero-dimensional sheaves lie in $\mathcal{P}(1)$.

Thus A is either torsion or torsion-free.

STEP 4. Suppose that A is torsion-free with a map $A \rightarrow \mathcal{O}_x$. A result of Mukai [12, Proposition 2.14] shows that A must be locally-free. Thus $\text{Hom}_X(A, \mathcal{O}_x) = \mathbb{C}^r$ for all $x \in X$ where r is the rank of A . Given $x \in X$ there is a short exact sequence in

$\mathcal{P}(1)$ of the form

$$0 \longrightarrow A' \longrightarrow \mathcal{O}_x \longrightarrow B \longrightarrow 0$$

where A' has all Jordan-Hölder factors isomorphic to A and $\mathrm{Hom}_X(A, B) = 0$. Since A is spherical (and hence has no self-extensions) it follows that $A' = A^{\oplus p}$ for some p , and considering the long exact sequence obtained by applying the functor $\mathrm{Hom}_X(A, -)$ shows that $p = r$ and that the map $A^{\oplus r} \rightarrow \mathcal{O}_x$ is the canonical evaluation map. It follows that $B = T_A(\mathcal{O}_x)$.

Suppose for a contradiction that B is not stable in σ . Lemma 12.2 shows that there is a stable spherical object $D \in \mathcal{P}(1)$ with either a nonzero map $D \rightarrow B$ or a nonzero map $B \rightarrow D$. The fact that σ is general implies that $v(D)$ lies in the lattice L . Since $v(B)^2 = -2 = v(A)^2$ and $v(\mathcal{O}_x)^2 = 0$, a quick calculation shows that the only possibilities are $v(D) = \pm v(A)$, and since $A, D \in \mathcal{P}(1)$ we must take the positive sign. The Riemann-Roch theorem then gives $\chi(A, D) = 2$, so that by Serre duality there is either a map $D \rightarrow A$ or a map $A \rightarrow D$. Since A and D are supposed to be stable of the same phase one concludes that $D = A$. But

$$\mathrm{Hom}_X(A, B) = \mathrm{Hom}_X(T_A^{-1}(A), \mathcal{O}_x) = \mathrm{Hom}_X(A[1], \mathcal{O}_x) = 0$$

and similarly $\mathrm{Hom}_X(B, A) = 0$ which gives a contradiction. So B is stable, and the alternative (A^+) applies.

STEP 5. Suppose instead that A is a torsion sheaf. The Riemann-Roch theorem shows that A is supported on a (-2) -curve $C \subset X$. If

$$0 \longrightarrow D \longrightarrow A \longrightarrow E \longrightarrow 0$$

is a short exact sequence of sheaves then Lemma 10.1(c) and Lemma 12.3 show that $D \in \mathcal{P}(0)$ and $E \in \mathcal{P}(1)$. Since σ is general $v(D)$ and $v(E)$ are in the lattice L . In particular it follows from this that C must be non-singular, since if $C' \subset C$ is an irreducible component there is a surjection of sheaves $A \rightarrow A|_{C'}$ which would lead to a non-trivial exact sequence as above. Riemann-Roch then shows that since A is spherical one has $A = \mathcal{O}_C(k+1)$ for some integer k . For any point $x \in C$ there is a unique map $A \rightarrow \mathcal{O}_x$ with cone $B = \mathcal{O}_C(k)[1]$. Since A is stable, this cone must also lie in $\mathcal{P}(1)$.

Suppose for a contradiction that B is not stable. If $D \in \mathcal{P}(1)$ is a stable factor then by Lemma 12.4 D is spherical and since σ is general one concludes as in Step 4 that $v(D)$ lies in the lattice L and hence $v(D) = \pm(0, C, l)$ for some l . Note that $Z(A)$ and $Z(B)$ lie on the negative real axis with $Z(A) + Z(B) = -1$. Also $Z(D)$ must lie on the negative real axis with $|Z(D)| < |Z(B)|$. The only possibility is that $v(D) = v(A)$ which implies $D = A$. But if all stable factors of D are isomorphic to A one has $v(D) = nv(A)$ which is clearly false. Thus $\mathcal{O}_C(k)[1]$ must be stable in σ and alternative (C_k) applies.

STEP 6. The cases when $A \in \mathcal{P}(1)$ with a map $\mathcal{O}_x \rightarrow A$ are dealt with by a similar analysis. For points $\tau = (W, \mathcal{Q}) \in W(X)$ sufficiently close to σ the object A is also stable in τ , and the nonzero map $\mathcal{O}_x \rightarrow A$ then implies that $A \in \mathcal{Q}(\phi)$ with $1 < \phi < 1 + \epsilon$. Lemma 10.1 then shows that there is a triangle

$$\begin{array}{ccc} D & \longrightarrow & A \\ & \swarrow \text{dashed} & \searrow \\ & E & \end{array}$$

with $D = H^{-2}(A)[2]$ and $E = H^{-1}(A)[1]$, and that moreover the sheaf $H^{-2}(A)$ is torsion-free. Applying the argument of Step 2 one concludes that one of D or E is zero.

If $E = 0$ then $A[-2]$ is a torsion-free sheaf and hence by Mukai's result [12, Proposition 2.14] locally-free. The argument of Step 4 then shows that one is in the situation (A^-) . If $D = 0$ then the argument of Step 4 shows that $A[-1]$ is either torsion or torsion-free, and the second possibility cannot hold because then $A[-1]$ would be locally-free and there could not be a nonzero map $\mathcal{O}_x \rightarrow A$. Thus $A[-1]$ is a torsion sheaf and the argument of Step 5 then shows that situation (C_k) holds for some k .

STEP 7. For the last statement, note that the triangles defining $T_A(\mathcal{O}_x)$ and $T_A^{-1}(\mathcal{O}_x)$ exist abstractly, so that a stability condition σ satisfies (A^-) precisely if the objects $A[2]$ and $T_A^{-1}(\mathcal{O}_x)$ are stable of the same phase for all $x \in X$. Similarly σ satisfies (A^+) precisely if A and $T_A(\mathcal{O}_x)$ are stable of the same phase for all $x \in X$. These two possibilities are clearly related by the equivalence T_A^2 . Similarly for the cases (C_k) where one notes that the short exact sequence of sheaves on \mathbb{P}^1

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(k-1) \longrightarrow \mathrm{Hom}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}(k), \mathcal{O}_{\mathbb{P}^1}(k+1)) \otimes \mathcal{O}_{\mathbb{P}^1}(k) \longrightarrow \mathcal{O}_{\mathbb{P}^1}(k+1) \longrightarrow 0$$

implies that $T_{\mathcal{O}_C(k)}(\mathcal{O}_C(k+1)) = \mathcal{O}_C(k-1)[1]$. \square

The proof used the following three simple results.

Lemma 12.2. *Suppose $\sigma = (Z, \mathcal{P}) \in \mathrm{Stab}(X)$ is a stability condition on X and $0 \neq E \in \mathcal{P}(1)$ is a semistable object of phase 1. If $\mathrm{Hom}_X^1(E, E) = 0$ then all stable factors of E are spherical objects. If $\mathrm{Hom}_X^1(E, E) = \mathbb{C}^2$ and E is not stable then there is a stable, spherical object $A \in \mathcal{P}(1)$ such that either $\mathrm{Hom}_X(A, E)$ or $\mathrm{Hom}_X(E, A)$ is nonzero.*

Proof. Consider Jordan-Hölder filtrations of E in the finite length category $\mathcal{P}(1)$. Take a stable object $F \in \mathcal{P}(1)$ with a nonzero map $F \rightarrow E$. Grouping factors together, there is a short exact sequence

$$0 \longrightarrow F' \longrightarrow E \longrightarrow G \longrightarrow 0$$

in $\mathcal{P}(1)$ such that all the Jordan-Hölder factors of F' are isomorphic to F , and $\mathrm{Hom}_X(F', G) = 0$. Applying Lemma 5.2 shows that

$$\dim_{\mathbb{C}} \mathrm{Hom}_X^1(F', F') + \dim_{\mathbb{C}} \mathrm{Hom}_X^1(G, G) \leq \dim_{\mathbb{C}} \mathrm{Hom}^1(E, E) \leq 2,$$

and since both spaces have even dimension at least one of them is zero. In the case $\mathrm{Hom}_X^1(E, E) = 0$ both spaces are zero, so $v(F)^2 < 0$ and hence F is spherical. Replacing E by G and repeating the argument gives the result.

Suppose instead that $\dim_{\mathbb{C}} \mathrm{Hom}_X^1(E, E) = 2$. If $\mathrm{Hom}_X^1(F', F') = 0$ then again F is spherical and setting $A = F$ gives the result. If $\mathrm{Hom}_X^1(G, G) = 0$ then all the stable factors of G are spherical, and so there is a stable, spherical object $A \in \mathcal{P}(1)$ with a nonzero map $G \rightarrow A$, and hence a nonzero map $E \rightarrow A$. \square

Lemma 12.3. *Suppose $\sigma = (Z, \mathcal{P})$ is a stability condition and $A \in \mathcal{P}(1)$ is stable. Given a triangle*

$$\begin{array}{ccc} D & \xrightarrow{\quad} & A \\ & \swarrow \text{dashed} & \searrow f \\ & E & \end{array}$$

with $D \in \mathcal{P}(\geq 0)$ and $E \in \mathcal{P}(\leq 1)$ and $f \neq 0$ then $D \in \mathcal{P}(0)$ and $E \in \mathcal{P}(1)$.

Proof. Considering the triangle $A \rightarrow E \rightarrow D[1]$ shows that $E \in \mathcal{P}(\geq 1)$ and hence $E \in \mathcal{P}(1)$. Since A is simple in the abelian category $\mathcal{P}(1)$ the map f is a monomorphism in $\mathcal{P}(1)$ so the quotient $D[1]$ also lies in $\mathcal{P}(1)$. \square

Lemma 12.4. *If $E \in \mathcal{D}(X)$ satisfies $\mathrm{Hom}_X^1(E, E) = 0$ then the same is true of each of its cohomology sheaves $H^i(E)$.*

Proof. There is a spectral sequence

$$E_2^{p,q} = \bigoplus_i \mathrm{Ext}_X^p(H^i(E), H^{i+q}(E)) \implies \mathrm{Hom}_X^{p+q}(E, E)$$

whose $E_2^{1,0}$ term survives to infinity. \square

13. PROOF OF THE MAIN THEOREM

It is now possible to use the results of the last two sections to prove Theorem 1.1. The crucial point is that the images of the closure of the set $U(X)$ under the elements of the group $\mathrm{Aut} \mathcal{D}(X)$ cover the entire connected component $\mathrm{Stab}^\dagger(X)$.

Lemma 13.1. *If $\sigma \in \mathrm{Stab}^\dagger(X)$ is a stability condition satisfying one of the assumptions A^+ , A^- or C_k of Theorem 12.1 then σ lies in the boundary of $U(X)$.*

Proof. Consider the A^+ case, the rest being similar. Set $B = T_A(\mathcal{O}_x)$. Then A and B are stable in σ of the same phase ϕ and there are short exact sequences

$$0 \longrightarrow A^{\oplus r} \longrightarrow \mathcal{O}_x \longrightarrow B \longrightarrow 0$$

in $\mathcal{P}(\phi)$. Applying an element of $\tilde{\text{GL}}^+(2, \mathbb{R})$ one can assume that $\phi = 1$.

Since $\text{Stab}^\dagger(X)$ is a good component and the objects $\{\mathcal{O}_x : x \in X\}$ have bounded mass, one can take a wall and chamber decomposition as in Proposition 9.3. Clearly σ must lie on a wall since all the objects \mathcal{O}_x are semistable but not stable. There exists at least one chamber C such that σ lies in the closure of C and such that $\text{Im } W(A)/W(B) < 0$ for stability conditions $(W, \mathcal{Q}) \in C$ close to σ . It will be enough to show that all \mathcal{O}_x are stable in C so that $C \subset U(X)$ and hence σ lies in the closure of $U(X)$.

Consider a stability condition $\tau = (W, \mathcal{Q}) \in C$ such that $f(\sigma, \tau) < \frac{1}{8}$. Let \mathcal{A} be the abelian subcategory $\mathcal{P}((\frac{1}{2}, \frac{3}{2}]) \subset \mathcal{D}(X)$. If \mathcal{O}_x is not stable there is a short exact sequence in \mathcal{A}

$$0 \longrightarrow D \longrightarrow \mathcal{O}_x \longrightarrow E \longrightarrow 0$$

with $\text{Im } W(D)/W(E) > 0$. One cannot have $\text{Im } Z(D)/Z(E) > 0$ because \mathcal{O}_x is semistable in σ . By the chamber structure one must have $Z(D)/Z(E) \in \mathbb{R}_{>0}$ so that $D, E \in \mathcal{P}(1)$ and the above sequence is a short exact sequence in $\mathcal{P}(1)$. Considering stable factors of \mathcal{O}_x and noting that $\text{Hom}_X(\mathcal{O}_x, A) = 0$ shows that all stable factors of D are equal to A so that $v(A) = nv(D)$. But this contradicts $\text{Im } W(A)/W(B) < 0$. \square

Proposition 13.2. *The connected component $\text{Stab}^\dagger(X) \subset \text{Stab}(X)$ is mapped by π onto the open subset $\mathcal{P}_0^+(X) \subset \mathcal{N}(X) \otimes \mathbb{C}$.*

Proof. The fact that $\pi(\text{Stab}^\dagger(X))$ contains $\mathcal{P}_0^+(X)$ follows from the fact that π is a covering map over $\mathcal{P}_0^+(X)$. The hard part is to prove the reverse inclusion.

Take a stability condition $\sigma \in \text{Stab}^\dagger(X)$. There is a continuous path $\gamma : [0, 1] \rightarrow \text{Stab}^\dagger(X)$ such that $\gamma(0) \in U(X)$ and $\sigma = \gamma(1)$. One can find a compact subset $B \subset \text{Stab}^\dagger(X)$ such that $\gamma([0, 1])$ lies in its interior. Let S be the set of objects E of $\mathcal{D}(X)$ such that $v(E) = v(\mathcal{O}_x)$ and E is semistable for some stability condition $\sigma \in B$. Then $m_\sigma(E) = |Z(E)| \leq m_\sigma(\mathcal{O}_x)$ and since B is compact it follows that S has bounded mass in the component $\text{Stab}^\dagger(X)$. Consider the corresponding wall and chamber structure as in Section 9.

Since π is a local homeomorphism, and the walls are locally-finite, one can deform γ a little so that there are real numbers $0 = t_0 < t_1 < \dots < t_n = 1$ such that each interval $I_i = (t_i, t_{i+1})$ is mapped by γ into one of the chambers, and such that each

point $\gamma(t_i)$ for $0 \leq i < n$ lies on only one wall. Thus $\gamma(I_0) \subset U(X)$ and $\gamma(t_1)$ is a general point of the boundary of $U(X)$.

It follows from Theorem 12.1 and Lemma 13.1 that at each general point σ of the boundary of $U(X)$ there is an autoequivalence Φ such that $\Phi(\sigma)$ also lies in the boundary of $U(X)$. Moreover these autoequivalences all preserve the class of \mathcal{O}_x in $\mathcal{N}(X)$. The important point is that these autoequivalences reverse the orientation of the boundary of $U(X)$.

Thus if $\sigma \in \partial U(X)$ is general of type (A^-) , then locally near σ the boundary of $U(X)$ is given by the real quadric $\text{Im } Z(A)/Z(\mathcal{O}_x) = 0$, and $U(X)$ is the side where $\text{Im } Z(A)/Z(\mathcal{O}_x) > 0$. Applying the equivalence T_A^2 gives a new stability condition on the boundary of $U(X)$, this time of type (A^+) . Locally at $T_A^2(\sigma)$ the boundary is still given by the equation $\text{Im } Z(A)/Z(\mathcal{O}_x) = 0$, but now $U(X)$ is on the side where $\text{Im } Z(A)/Z(\mathcal{O}_x) < 0$.

On the other hand, if $\sigma \in \partial U(X)$ is general of type (C_k) , then the boundary of $U(X)$ is given locally by $\text{Im } Z(\mathcal{O}_C)/Z(\mathcal{O}_x) = 0$, with $U(X)$ being the side where $\text{Im } Z(\mathcal{O}_C)/Z(\mathcal{O}_x) > 0$. Applying $T_{\mathcal{O}_C(k)}$ identifies the (C_k) part of the boundary of $U(X)$ with the (C_{k-1}) part, and the fact that $T_{\mathcal{O}_C(k)}$ acts on $\mathcal{N}(X)$ via a reflection shows that again this identification is orientation-reversing.

Thus there is an autoequivalence $\Phi_1 \in \text{Aut } \mathcal{D}(X)$, preserving the class of \mathcal{O}_x in $\mathcal{N}(X)$, such that $\Phi_1(\gamma(t_1))$ lies in the boundary of $U(X)$, and for points $t > t_1$ close to t one has $\Phi_1(\gamma(t)) \in U(X)$, which is to say $\Phi_1^{-1}(\mathcal{O}_x)$ is stable in $\gamma(t)$. By the chamber structure it follows that $\Phi_1(I_1) \subset U(X)$. Repeating the argument shows that there is an autoequivalence $\Phi \in \text{Aut } \mathcal{D}(X)$ preserving the class of \mathcal{O}_x such that $\Phi(\sigma)$ lies in the closure of $U(X)$.

By Corollary 11.3, the fact that $\Phi(\sigma)$ lies in the closure of $U(X)$ implies that the real and imaginary parts of $\pi(\sigma) \in \mathcal{N}(X) \otimes \mathbb{C}$ spans a non-negative two-plane in $\mathcal{N}(X) \otimes \mathbb{R}$. Since this holds for any stability condition in $\text{Stab}^\dagger(X)$, and the map π is open on a full component of $\text{Stab}(X)$, it follows that $\pi(\text{Stab}^\dagger(X)) \subset \mathcal{P}(X)$. Since $\text{Stab}^\dagger(X)$ is connected the image must in fact lie in $\mathcal{P}^+(X)$.

Finally, suppose that $\sigma = (Z, \mathcal{P}) \in \text{Stab}^\dagger(X)$ satisfies $Z(\delta) = 0$ for some class $\delta \in \Delta(X)$. By Corollary 11.3 one can assume that σ lies in the boundary of $U(X)$. According to Proposition 9.3 the boundary of $U(X)$ in a neighbourhood of σ is made up of a finite union of codimension one submanifolds of $\text{Stab}(X)$ each passing through σ . By Theorem 12.1, each of these components is of the form $Z(A)/Z(\mathcal{O}_x) \in \mathbb{R}_{>0}$ for some spherical object A . Moreover each A is stable of the same phase as \mathcal{O}_x at a general point of the corresponding boundary component, and hence semistable at σ .

Fix one component Γ of the boundary of $U(X)$ near σ and suppose δ lies in the corresponding rank two sublattice $L \subset \mathcal{N}(X)$ spanned by $v(A)$ and $v(\mathcal{O}_x)$. If $r(A) > 0$ then since $v(A)$ and δ both lie in $\Delta(X)$, one has $v(A) = \pm\delta$, which is impossible because A is semistable at σ whereas $Z(\delta) = 0$. If $r(A) = 0$ then the component Γ is of the form (C_k) for some (-2) -curve $C \subset X$ and some integer k , so that at a general point the Jordan-Hölder filtration of \mathcal{O}_x for points $x \in C$ are of the form

$$0 \longrightarrow \mathcal{O}_C(k+1) \longrightarrow \mathcal{O}_x \longrightarrow \mathcal{O}_C(k)[1] \longrightarrow 0.$$

Then $\mathcal{O}_C(k)$ and $\mathcal{O}_C(k+1)$ are at least semistable at σ so have non-vanishing central charge, and hence $0 < |Z(\mathcal{O}_C(k))| < |Z(\mathcal{O}_x)|$. But if $\delta \in L$ then $\delta = \pm v(\mathcal{O}_C(l))$ for some integer l and then $Z(\delta) = 0$ implies that $Z(\mathcal{O}_C(k)) = (k-l)Z(\mathcal{O}_x)$ which gives a contradiction.

Since the image of $\text{Stab}^\dagger(X)$ is open in $\mathcal{N}(X) \otimes \mathbb{C}$ it now follows that there exist stability conditions in $\text{Stab}^\dagger(X)$ arbitrarily close to σ which are contained in $U(X)$ and which still satisfy $Z(\delta) = 0$. But this is impossible by Corollary 11.3. \square

The following is a restatement of Theorem 1.1.

Theorem 13.3. *The map $\pi: \text{Stab}^\dagger(X) \rightarrow \mathcal{P}_0^+(X)$ is a covering map. The subgroup of $\text{Aut}^0 \mathcal{D}(X)$ fixing the connected component $\text{Stab}^\dagger(X) \subset \text{Stab}(X)$ acts freely on $\text{Stab}^\dagger(X)$ and is the group of deck transformations.*

Proof. The fact that π is a covering map is Proposition 8.3. It is clear that the given subgroup of $\text{Aut}^0 \mathcal{D}(X)$ acts on $\text{Stab}^\dagger(X)$ preserving the map π . What we are required to show is that given stability conditions σ and τ in $\text{Stab}^\dagger(X)$ with the same central charge there is a unique $\Phi \in \text{Aut}^0 \mathcal{D}(X)$ with $\Phi(\tau) = \sigma$. Since π is a covering map, it is enough to check this for a fixed $\sigma \in \text{Stab}^\dagger(X)$ which we may as well assume lies in $U(X)$.

To prove uniqueness suppose $\Phi \in \text{Aut}^0 \mathcal{D}(X)$ is such that $\Phi(\sigma) = \sigma$. Thus if E is stable in σ of a given phase then so is $\Phi(E)$. By Lemma 10.1, the only objects $E \in \mathcal{D}(X)$ which are stable in σ with Mukai vector $v(E) = v(\mathcal{O}_x)$ are the skyscraper sheaves \mathcal{O}_x themselves together with their even shifts. It follows that Φ takes skyscrapers to skyscrapers which implies that $\Phi(E) = f^*(E \otimes L)$ for some $L \in \text{Pic}(X)$ and some $f \in \text{Aut}(X)$. Since $\Phi \in \text{Aut}^0 \mathcal{D}(X)$ it follows (using the Torelli theorem) that Φ is the identity.

Now assume that σ and τ have the same central charge and $\sigma \in U(X)$. By the argument of Proposition 13.2, there is an autoequivalence $\Phi \in \text{Aut} \mathcal{D}(X)$ such that $\Phi(\tau)$ lies in the closure of $U(X)$, so moving σ a bit one can assume that σ and $\Phi(\tau)$ both lie in $U(X)$. Moreover one can assume that Φ preserves the class of \mathcal{O}_x in

$\mathcal{N}(X)$. Composing Φ with a twist by a line bundle, which preserves the set $U(X)$, one can assume that Φ also preserves the class of \mathcal{O}_X . Thus the action of Φ on $H^*(X, \mathbb{Z})$ preserves the decomposition

$$H^*(X, \mathbb{Z}) = H^0(X, \mathbb{Z}) \oplus H^2(X, \mathbb{Z}) \oplus H^4(X, \mathbb{Z}).$$

Since σ and $\Phi(\tau)$ both lie in $U(X)$, the induced Hodge isometry of $H^2(X, \mathbb{Z})$ is effective [1, Proposition VIII.3.10]. It follows from the Torelli theorem [1, Theorem VIII.11.1] that composing Φ with an automorphism of X , which again preserves $U(X)$, one can assume that Φ acts trivially on $H^*(X, \mathbb{Z})$. But now σ and $\Phi(\tau)$ have the same central charge and both lie in $U(X)$. It follows from Corollary 11.3 that composing Φ with an even shift one has $\Phi(\tau) = \sigma$ which completes the proof. \square

14. THE LARGE VOLUME LIMIT

The aim of this section is to study the class of stable objects in the stability condition $\sigma \in V(X)$ corresponding to a point $\exp(\beta + i\omega) \in \mathcal{P}_0^+(X)$ in the limit as $\omega \rightarrow \infty$. This is what physicists would refer to as a large volume limit, and string theory predicts that the BPS branes in this limit are just Gieseker stable sheaves. In fact, in the presence of a nonzero B-field, Gieseker stability gets twisted. This leads to a notion of stability first introduced by Matsuki and Wentworth [10].

Throughout this section $\beta, \omega \in \text{NS}(X) \otimes \mathbb{R}$ will be a fixed pair of \mathbb{R} -divisor classes with $\omega \in \text{Amp}(X)$ ample. Given a torsion-free sheaf E on X with Mukai vector $v(E) = (r(E), c_1(E), s(E))$ define

$$\mu_{\beta, \omega}(E) = \frac{(c_1(E) - r(E)\beta) \cdot \omega}{r(E)} \quad \text{and} \quad \nu_{\beta, \omega}(E) = \frac{s(E) - c_1(E) \cdot \beta}{r(E)}.$$

Note that $\mu_{\beta, \omega}(E) = \mu_{\omega}(E) - \beta \cdot \omega$. The following definition reduces to Gieseker stability in the case $\beta = 0$.

Definition 14.1. A torsion-free sheaf E on X is said to be twisted semistable with respect to the pair (β, ω) if

$$\mu_{\beta, \omega}(A) < \mu_{\beta, \omega}(E) \quad \text{or} \quad (\mu_{\beta, \omega}(A) = \mu_{\beta, \omega}(E) \quad \text{and} \quad \nu_{\beta, \omega}(A) \leq \nu_{\beta, \omega}(E))$$

for all subsheaves $0 \neq A \subset E$.

Note that a twisted semistable sheaf is, in particular, slope semistable. The mathematical reason for introducing twisted stability is that unlike slope stability, Gieseker stability is not preserved by twisting by line bundles. Thus if $\beta \in \text{NS}(X)$ is the first Chern class of a line bundle L , then a torsion-free sheaf E is Gieseker semistable with respect to ω if and only if $E \otimes L$ is twisted semistable with respect to the pair (β, ω) . The above definition just generalises this idea to arbitrary elements $\beta \in \text{NS}(X) \otimes \mathbb{R}$.

Proposition 14.2. *Fix a pair $\beta, \omega \in \text{NS}(X) \otimes \mathbb{Q}$ with $\omega \in \text{Amp}(X)$ ample. For integers $n \gg 0$ there is a unique stability condition $\sigma_n \in V(X)$ satisfying $\pi(\sigma_n) = \exp(\beta + in\omega)$. Suppose $E \in \mathcal{D}(X)$ satisfies*

$$r(E) > 0 \quad \text{and} \quad (c_1(E) - r(E)\beta) \cdot \omega > 0.$$

Then E is semistable in σ_n for all $n \gg 0$ precisely if E is a shift of a (β, ω) -twisted semistable sheaf on X .

Proof. Providing $(n\omega)^2 > 2$ one has $\exp(\beta + in\omega) \in \mathcal{L}(X)$ so it follows from Proposition 11.2 that there is a unique stability condition $\sigma_n \in V(X)$ satisfying $\pi(\sigma_n) = \exp(\beta + in\omega)$. Note that each of the stability conditions $\sigma_n = (Z_n, \mathcal{P}_n)$ has the same heart $\mathcal{A}(\beta, \omega) = \mathcal{P}_n((0, 1])$. Note also that if E is a nonzero sheaf on X then equation (\star) of Section 6 shows that

$$\lim_{n \rightarrow \infty} \frac{1}{\pi} \arg Z_n(E) = \begin{cases} 0 & \text{if } \text{supp}(E) = X, \\ \frac{1}{2} & \text{if } \dim \text{supp}(E) = 1, \\ 1 & \text{if } \text{supp}(E) = 0. \end{cases}$$

Take an object $E \in \mathcal{D}(X)$ with $r(E) > 0$ and $\mu_{\beta, \omega}(E) > 0$. First suppose that E is semistable in σ_n for all $n \gg 0$. Applying a shift one may as well assume that E lies in $\mathcal{A}(\beta, \omega)$. According to Lemma 10.1, E has non-vanishing cohomology sheaves in just two degrees, and there is a short exact sequence in $\mathcal{A}(\beta, \omega)$

$$0 \longrightarrow H^{-1}(E)[1] \longrightarrow E \longrightarrow H^0(E) \longrightarrow 0.$$

Now $H^{-1}(E)$ is a torsion-free sheaf, so according to the asymptotic formula above, the phase of E tends to 0 in the limit $n \rightarrow \infty$, whereas, if the object $H^{-1}(E)[1]$ is nonzero, its phase must tend to 1. Since E is semistable in σ_n for $n \gg 0$ it follows that $H^{-1}(E) = 0$ and hence E is a sheaf. A similar argument with the asymptotic formula shows that E must be torsion-free. Note that the $\mu_{\beta, \omega}$ -semistable factors of E all have positive slope.

Suppose E is not (β, ω) -twisted semistable. Then there is a destabilising sequence

$$0 \longrightarrow A \longrightarrow E \longrightarrow B \longrightarrow 0$$

of sheaves on X such that A and B lie in $\mathcal{A}(\beta, \omega)$ and A is a μ_{ω} -semistable sheaf with $\mu_{\beta, \omega}(A) > \mu_{\beta, \omega}(E)$. Rewriting equation (\star) of Section 6 gives

$$\frac{Z_n(E)}{r(E)} - \frac{Z_n(A)}{r(A)} = -(\nu_{\beta, \omega}(E) - \nu_{\beta, \omega}(A)) + in(\mu_{\beta, \omega}(E) - \mu_{\beta, \omega}(A)).$$

Since the phases of A and E tend to zero, this implies $\arg Z_n(A) > \arg Z_n(E)$ for all $n \gg 0$ which contradicts semistability of E . Thus E is (β, ω) -twisted semistable.

For the converse, suppose that E is a (β, ω) -twisted semistable torsion-free sheaf with $\mu_{\beta, \omega}(E) > 0$. In particular, E is μ_ω -semistable so that $E \in \mathcal{A}(\beta, \omega)$. Suppose

$$0 \longrightarrow A \longrightarrow E \longrightarrow B \longrightarrow 0$$

is a short exact sequence in $\mathcal{A}(\beta, \omega)$. Taking cohomology gives a long exact sequence of sheaves

$$(\dagger) \quad 0 \longrightarrow H^{-1}(B) \longrightarrow A \longrightarrow E \longrightarrow H^0(B) \longrightarrow 0.$$

By definition of the category $\mathcal{A}(\beta, \omega)$ the sheaf $H^{-1}(B)$ has slope $\mu_{\beta, \omega} \leq 0$ whereas A has slope $\mu_{\beta, \omega} > 0$. Since E is semistable it follows that $\mu_{\beta, \omega}(A) \leq \mu_{\beta, \omega}(E)$ and if equality holds then $\nu_{\beta, \omega}(A) \leq \nu_{\beta, \omega}(E)$. Lemma 14.3 shows that the set of possible values of $\nu_{\beta, \omega}(A)$ is bounded above. The phase of E tends to zero as $n \rightarrow \infty$ and moreover the real part of $Z_n(E)$ tends to $+\infty$. It follows that $\arg Z_n(A) \leq \arg Z_n(E)$ for all subobjects $A \subset E$ in $\mathcal{A}(\beta, \omega)$ and all $n \gg 0$. Thus E is semistable in σ_n for $n \gg 0$. \square

Lemma 14.3. *Let E be a (β, ω) -twisted semistable torsion-free sheaf with $\mu_{\beta, \omega}(E) > 0$. Then the set of values of $\nu_{\beta, \omega}(A)$ as A ranges through all nonzero subobjects of E in $\mathcal{A}(\beta, \omega)$ is bounded above.*

Proof. Since β is rational, Theorem 5.3 shows that there exist torsion-free μ_ω -semistable sheaves P with Mukai vector $(r, r\beta, s)$ for some $s \in \mathbb{Z}$ and some $r > 0$. Decomposing into stable factors and taking a double dual one may assume that P is in fact μ_ω -stable and locally-free.

The Riemann-Roch formula gives

$$\chi(P, A) = s(P)r(A) + r(P)(s(A) - c_1(A) \cdot \beta).$$

Comparing with the formula for $\nu_{\beta, \omega}(A)$ shows that it is enough to give an upper bound for $\chi(P, A)/r(A)$. Consider the exact sequence (\dagger) and put $D = H^{-1}(B)$. Since A has Harder-Narasimhan factors of positive slope $\mu_{\beta, \omega}$ there can be no maps $A \rightarrow P$. Thus it suffices to bound the quotient $\dim_{\mathbb{C}} \text{Hom}_X(P, D)/r(D)$.

Since the Harder-Narasimhan factors of D have non-positive slope $\mu_{\beta, \omega}$, any nonzero map $f: P \rightarrow D$ is an injection with torsion-free quotient. Indeed, if f is not injective, then since P is $\mu_{\beta, \omega}$ -stable, the image of f would have strictly positive slope $\mu_{\beta, \omega}$ which is impossible. Thus there is a short exact sequence

$$0 \longrightarrow P \longrightarrow D \longrightarrow Q \longrightarrow 0.$$

If Q has torsion subsheaf T , then the induced map $D \rightarrow Q/T$ has kernel K fitting into an exact sequence

$$0 \longrightarrow P \longrightarrow K \longrightarrow T \longrightarrow 0.$$

If T is supported in dimension one then K has strictly positive slope $\mu_{\beta,\omega}$ which again is impossible. But if T is supported in dimension zero then the above sequence splits which is impossible since D is torsion-free. Hence $T = 0$.

Replacing D by Q and proceeding by induction on $r(D)$ it is easy to see that

$$\dim_{\mathbb{C}} \operatorname{Hom}_X(P, D) \leq r(D)/r(P).$$

which completes the proof. \square

15. STABILITY CONDITIONS ON ABELIAN SURFACES

Suppose that instead of a K3 surface one considers an abelian surface X . The theory developed in this paper carries over virtually unchanged to this case, because the only features of a K3 surface which were used were homological properties such as the Riemann-Roch theorem or Serre duality. The most important difference between abelian surfaces and K3 surfaces from the point of view of this paper is the following

Lemma 15.1. *If X is an abelian surface then $\mathcal{D}(X)$ has no spherical objects.*

Proof. In fact there are no nonzero objects $E \in \mathcal{D}(X)$ with $\operatorname{Hom}_X^1(E, E) = 0$. The basic reason is that there can be no rigid objects on a torus because of the continuous automorphism group.

By Lemma 12.4 it is enough to consider the case when E is a sheaf. Note that if E is a vector bundle, then \mathcal{O}_X is a direct summand of $E^\vee \otimes E$, so that

$$\mathbb{C}^2 = H^1(X, \mathcal{O}_X) \subset H^1(X, E^\vee \otimes E) = \operatorname{Ext}_X^1(E, E).$$

For a general sheaf E one can twist by a sufficiently ample line bundle and apply the Fourier-Mukai transform [11] to obtain a vector bundle with the same Ext-algebra. \square

Suppose then that X is an abelian surface over \mathbb{C} . Consider the even cohomology lattice

$$H^{2*}(X, \mathbb{Z}) = H^0(X, \mathbb{Z}) \oplus H^2(X, \mathbb{Z}) \oplus H^4(X, \mathbb{Z}),$$

equipped with the Mukai bilinear form. This is an even and non-degenerate lattice of signature $(4, 4)$. The Todd class of X is trivial, so the Mukai vector of an object $E \in \mathcal{D}(X)$ is the triple

$$v(E) = \operatorname{ch}(E) = (r(E), c_1(E), \operatorname{ch}_2(E)) \in \mathcal{N}(X) = \mathbb{Z} \oplus \operatorname{NS}(X) \oplus \mathbb{Z}.$$

As before, $\mathcal{N}(X)$ has signature $(2, \rho)$ and $\mathcal{P}^+(X) \subset \mathcal{N}(X) \otimes \mathbb{C}$ is defined to be one component of the set of vectors $\tilde{U} \in \mathcal{N}(X) \otimes \mathbb{C}$ which span positive definite two-planes.

It is a simple consequence of Lemma 15.1 that, up to a shift, every autoequivalence $\Phi \in \operatorname{Aut} \mathcal{D}(X)$ takes skyscraper sheaves to sheaves [3, Corollary 2.10]. It

follows immediately that $\text{Aut}^0 \mathcal{D}(X)$ is generated by the double shift [2], together with twists by elements of $\text{Pic}^0(X)$, and pull-backs by automorphisms of X acting trivially on $H^*(X, \mathbb{Z})$. It also follows that no element of $\text{Aut} \mathcal{D}(X)$ exchanges the two components of $\mathcal{P}(X)$.

As before one can define an open subset $U(X) \subset \text{Stab}(X)$. This time however, because of the lack of spherical objects, $U(X) = \text{Stab}^\dagger(X)$ is actually a connected component of $\text{Stab}(X)$. Thus up to the action of $\text{GL}^+(2, \mathbb{R})$, every stability condition in $\text{Stab}^\dagger(X)$ is obtained from the construction of Section 6.

Note that twists by elements of $\text{Pic}^0(X)$ and pull-backs by automorphisms of X acting trivially on $H^*(X, \mathbb{Z})$ take skyscrapers to skyscrapers, so these elements of $\text{Aut}^0 \mathcal{D}(X)$ act trivially on $\text{Stab}^\dagger(X)$. Note also that $\mathcal{P}^+(X)$ is a $\text{GL}^+(2, \mathbb{R})$ -bundle over the contractible space

$$\{\beta + i\omega \in \text{NS}(X) \otimes \mathbb{C} : \omega^2 > 0\}$$

so that $\pi_1 \mathcal{P}^+(X) = \mathbb{Z}$. Putting these observations together gives

Theorem 15.2. *Let X be an abelian surface over \mathbb{C} . There is a connected component $\text{Stab}^\dagger(X) \subset \text{Stab}(X)$ which is mapped by π onto the open subset $\mathcal{P}^+(X) \subset \mathcal{N}(X) \otimes \mathbb{C}$. Moreover, the induced map*

$$\pi: \text{Stab}^\dagger(X) \longrightarrow \mathcal{P}^+(X)$$

is the universal cover, and the group of deck transformations is generated by the double shift functor [2].

One can also show directly that the action of the group $\text{Aut} \mathcal{D}(X)$ on $\text{Stab}(X)$ preserves the connected component $\text{Stab}^\dagger(X)$. Thus the analogue of Conjecture 1.2 holds in the abelian surface case.

REFERENCES

- [1] W. Barth, C. Peters and A. Van de Ven, Compact complex surfaces. Ergebnisse der Mathematik (3), 4. Springer-Verlag, Berlin, 1984.
- [2] A. Beilinson, J. Bernstein and P. Deligne, Faisceaux Pervers, Astérisque 100, Soc. Math de France (1983).
- [3] T. Bridgeland and A. Maciocia, Complex surfaces with equivalent derived categories, Math. Z. 236 (2001).
- [4] T. Bridgeland and A. Maciocia, Fourier-Mukai transforms for $K3$ and elliptic fibrations. J. Algebraic Geom. 11 (2002), no. 4.
- [5] T. Bridgeland, Stability conditions on triangulated categories, to appear in Ann. Math. (2007) Preprint math.AG/0212237.
- [6] T. Bridgeland, Spaces of stability conditions, Preprint math.AG/0611510.

- [7] M.R. Douglas, Dirichlet branes, homological mirror symmetry, and stability. Proceedings of the International Congress of Mathematicians, Vol. III (Beijing, 2002), 395–408, Higher Ed. Press, Beijing, 2002.
- [8] S.I. Gelfand and Yu. I. Manin, *Methods of Homological Algebra*, Springer-Verlag (1996).
- [9] D. Happel, I. Reiten, S.O. Smalø, Tilting in abelian categories and quasitilted algebras. *Mem. Amer. Math. Soc.* 120 (1996), no. 575.
- [10] K. Matsuki, R. Wentworth, Mumford-Thaddeus principle on the moduli space of vector bundles on an algebraic surface. *Internat. J. Math.* 8 (1997), no. 1, 97–148.
- [11] S. Mukai, Duality between $\mathcal{D}(X)$ and $\mathcal{D}(\hat{X})$ with its application to Picard sheaves, *Nagoya Math. J.* **81** (1981) 153–175.
- [12] S. Mukai, On the moduli space of bundles on K3 surfaces I, in: *Vector Bundles on Algebraic Varieties*, M.F. Atiyah et al., Oxford University Press (1987), 341–413.
- [13] S. Mukai, Fourier functor and its application to the moduli of bundles on an abelian variety. *Algebraic geometry, Sendai, 1985*, 515–550, *Adv. Stud. Pure Math.*, 10 (1987).
- [14] D.O. Orlov, Equivalences of derived categories and K3 surfaces, *J. Math. Sci. (New York)* 84 5 (1997) 1361–1381.
- [15] P. Seidel and R.P. Thomas, Braid group actions on derived categories of coherent sheaves, *Duke Math. J.* 108 (2001), no. 1, 37–108.
- [16] B. Szendrői, Diffeomorphisms and families of Fourier–Mukai transforms in mirror symmetry, *Applications of algebraic geometry to coding theory, physics and computation*, 317–337, NATO Sci. Ser., Kluwer 2001.
- [17] K. Yoshioka, Chamber structure of polarizations and the moduli of stable sheaves on a ruled surface, *Internat. J. Math.* 7 (1996), no. 3, 411–431.
- [18] K. Yoshioka, Moduli spaces of stable sheaves on abelian surfaces, *Math. Ann.* 321 (2001), no. 4, 817–884.

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