

FOURIER-MUKAI TRANSFORMS FOR ELLIPTIC SURFACES

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ABSTRACT. We compute a large number of moduli spaces of stable bundles on a general algebraic elliptic surface using a new class of relative Fourier-Mukai transforms.

1. INTRODUCTION

In recent years moduli spaces of stable bundles on projective surfaces have been extensively studied. Although various general results are known, few of these spaces have been described explicitly. One important development was S. Mukai's discovery [11] of a transform which allowed him to compute some of the moduli spaces on Abelian varieties [14]. This technique has recently been extended to cover K3 surfaces [4], [13], [17]. Here we introduce similar transforms for elliptic surfaces.

Much is already known about moduli of rank 2 bundles on elliptic surfaces thanks to work by R. Friedman [7], [8], amongst others. This research has led to several important results, including the smooth classification of elliptic surfaces. In this paper we use Mukai's techniques to study bundles of higher rank.

1.1. Let $X \xrightarrow{\pi} C$ be a relatively minimal algebraic elliptic surface over \mathbb{C} . Given a sheaf E on X we write its Chern class as a triple

$$(r(E), c_1(E), c_2(E)) \in \mathbb{Z} \times \text{NS}(X) \times \mathbb{Z}.$$

Here $\text{NS}(X)$ is the Néron-Severi group of X , i.e. the subgroup of $H^2(X, \mathbb{Z})$ generated by the Chern classes of line bundles on X . We denote the element of $\text{NS}(X)$ corresponding to a fibre of π by f .

Let (r, Λ, k) be a triple as above, and assume that $r > 1$ is coprime to $\Lambda \cdot f$. Then, as Friedman observed, there exist polarizations of X with respect to which a torsion-free sheaf E of Chern class (r, Λ, k) is stable iff the restriction of E to the general fibre of π is stable. For these polarizations, and sheaves of this Chern class, the notions of Gieseker stability, μ -semi-stability and μ -stability all coincide. Taking such a polarization, we define $\mathcal{M} = \mathcal{M}_X(r, \Lambda, k)$ to be the moduli space of stable torsion-free sheaves on X of Chern class (r, Λ, k) .

An argument of Friedman's shows that the projective scheme \mathcal{M} is smooth of dimension $\dim(\text{Pic}^\circ(X)) + 2t$, where

$$(1.1.1) \quad 2t = 2rk - (r-1)\Lambda^2 - (r^2-1)\chi(\mathcal{O}_X).$$

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If $t < 0$ then \mathcal{M} is empty, so we shall assume that t is non-negative. Our main result (Theorem 1.1 below) states that \mathcal{M} is irreducible and birationally equivalent to $\text{Pic}^\circ(Y) \times \text{Hilb}^t(Y)$, where Y is another elliptic surface over C .

1.2. Let us define λ_X to be the highest common factor of the fibre degrees of sheaves on X . Equivalently, λ_X is the smallest positive integer such that $X \xrightarrow{\pi} C$ has a holomorphic λ_X -multisection.

Given a pair of integers $a > 0$ and b such that $a\lambda_X$ is coprime to b , we define an elliptic surface $J_X(a, b)$ over C , whose fibre over a point $p \in C$ is canonically identified with (a component of) the moduli space of rank a , degree b , stable sheaves on the fibre X_p .

To do this, take a polarization of X of fibre degree coprime to b , and let $\mathcal{M}(X/C) \rightarrow C$ denote the relative moduli space of stable pure-dimension 1 sheaves on the fibres of π (see [18]). Then define $J_X(a, b)$ to be the union of those components of $\mathcal{M}(X/C)$ which contain a rank a , degree b vector bundle supported on a non-singular fibre of π .

The details of this construction, and the proof that $J_X(a, b)$ is an elliptic surface, are given in section 4. The essential point is that each component of the moduli of stable sheaves on an elliptic curve is again an elliptic curve, so that $J_X(a, b)$ has a natural elliptic fibration structure.

Theorem 1.1. *The moduli space $\mathcal{M} = \mathcal{M}_X(r, \Lambda, k)$ is a smooth (non-empty) projective variety and is birationally equivalent to*

$$\text{Pic}^\circ(J_X(a, b)) \times \text{Hilb}^t(J_X(a, b)),$$

where (a, b) is the unique pair of integers satisfying $br - a(\Lambda \cdot f) = 1$ and $0 < a < r$. Furthermore, if $r > a$ at the birational equivalence extends to give an isomorphism of varieties.

This generalises a result of Friedman who looked at the rank 2 moduli. In that case one always has $a = 1$, and the surface $J_X(a, b)$ is the relative Picard scheme $J^b(X)$ of [7].

1.3. To prove Theorem 1.1 we use a relative version of the Fourier-Mukai transform. Let $D(S)$ denote the bounded derived category of coherent sheaves on a Noetherian scheme S . In general, given two smooth varieties X and Y and an object \mathcal{P} of $D(X \times Y)$, one defines a functor $\Phi_{Y \rightarrow X}^{\mathcal{P}} : D(Y) \rightarrow D(X)$, by the formula

$$\Phi_{Y \rightarrow X}^{\mathcal{P}}(-) = \mathbf{R}\pi_{X,*}(\mathcal{P} \otimes^{\mathbf{L}} \pi_Y^*(-)),$$

where π_X and π_Y are the projections from $X \times Y$ to X and Y respectively. In some very special circumstances this functor is an equivalence of categories, and one then has a useful tool for studying sheaves on X and Y . Examples include the original Fourier-Mukai transform [11] (where X is an Abelian variety and Y is its dual) and the generalised transforms of [4] and [17] (where X and Y are isogenous K3 surfaces).

Theorem 1.2. *Given integers $a > 0$ and b with $a\lambda_X$ coprime to b , let $Y = J_X(a, b)$. Then there exist tautological sheaves on $X \times Y$, supported on*

$X \times_{\mathbb{C}} Y$, and for each such sheaf \mathcal{P} , the functor $\Phi_{Y \rightarrow X}^{\mathcal{P}} : D(Y) \rightarrow D(X)$ is an equivalence of categories.

Although this result involves derived categories in an essential way, we shall see that the functors $\Phi_{Y \rightarrow X}^{\mathcal{P}}$ often take sheaves to sheaves, and thus yield concrete results about moduli of vector bundles. Indeed, once the basic properties of these functors are known, the proof of Theorem 1.1 is relatively simple, and it seems likely that the same functors will prove useful for solving other moduli problems on elliptic surfaces. One might also expect similar results for higher-dimensional elliptic fibrations.

Notation. All schemes will be Noetherian \mathbb{C} -schemes and all morphisms will be morphisms over \mathbb{C} . By a sheaf on a scheme X we mean a coherent \mathcal{O}_X -module. A point of X will mean a closed point. $D(X)$ denotes the bounded derived category of sheaves on X . We refer to [9] for properties of $D(X)$.

Given an object E of $D(X)$ let $\mathcal{H}^i(E)$ denote the i th cohomology sheaf of E . We say that E is a sheaf if $\mathcal{H}^i(E) = 0$ when $i \neq 0$. The derived dual, $\mathbf{R}\mathcal{H}om(E, \mathcal{O}_X)$, is denoted E^\vee , and $E[n]$ denotes the object E shifted to the left by n places.

By a variety we mean an integral, separated scheme of finite type over \mathbb{C} . The canonical bundle of a smooth variety X is written ω_X , and for objects E and F of $D(X)$ we define

$$\chi(E, F) = \sum (-1)^i \dim_{\mathbb{C}} \mathrm{Hom}_{D(X)}(E, F[i]),$$

and write $\mathrm{ch}(E)$ for the Chern character of E . This defines a map

$$\mathrm{ch} : D(X) \rightarrow H^{2*}(X, \mathbb{C}).$$

We denote the image (an Abelian group) by $\mathrm{ch}(X)$.

We refer to [18] for the definitions of pure-dimension sheaves and stable sheaves on a projective scheme X . They depend on the polarization chosen for X .

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2. GENERAL PROPERTIES OF $\Phi_{Y \rightarrow X}^{\mathcal{P}}$

Throughout this section X and Y are smooth varieties and \mathcal{P} is an object of $D(X \times Y)$. We shall be mainly interested in the case when \mathcal{P} is a sheaf on $X \times Y$, flat over both factors. Let Φ denote the corresponding functor $\Phi_{Y \rightarrow X}^{\mathcal{P}} : D(Y) \rightarrow D(X)$ defined in the introduction. We state various properties of Φ which we shall need. These all appear in some form in Mukai's original papers on Abelian varieties and K3 surfaces [11], [13], [14]. The general results have been worked out by A. I. Bondal and D. O. Orlov [6], [17], and A. Maciocia [10].

2.1. First we define WIT (weak index theorem) sheaves. Given an object E of $D(Y)$ put

$$\Phi^i(E) = \mathcal{H}^i(\Phi(E)).$$

A sheaf E on Y is said to be Φ -WIT $_i$ if $\Phi^j(E) = 0$ for all $j \neq i$, or equivalently if $\Phi(E)[i]$ is a sheaf. We say E is Φ -WIT if it is Φ -WIT $_i$ for some i , and in this case we often write \hat{E} for $\Phi^i(E)$, and refer to \hat{E} as the *transform* of E .

2.2. By Grothendieck's Riemann-Roch theorem there is a group homomorphism $\text{ch}(\Phi)$ making the following diagram commute

$$\begin{array}{ccc} D(Y) & \xrightarrow{\Phi} & D(X) \\ \text{ch} \downarrow & & \text{ch} \downarrow \\ \text{ch}(Y) & \xrightarrow{\text{ch}(\Phi)} & \text{ch}(X) \end{array}$$

It is given by

$$\text{ch}(\Phi)(y) = \pi_{X,*}(p \cdot \pi_Y^* y),$$

where $p = \text{ch}(\mathcal{P}) \cdot \pi_Y^*(\text{td}_Y)$ and td_Y is the Todd class of Y .

2.3. Define

$$\mathcal{Q} = (\mathcal{P}^\vee \otimes \pi_X^* \omega_X)[\dim X + \dim Y - \dim \mathcal{P}],$$

and put $\Psi = \Phi_{X \rightarrow Y}^{\mathcal{Q}}$. It is a simple consequence of Grothendieck-Verdier duality (see e.g. [6], Lemma 1.2) that $\Psi[\dim \mathcal{P} - \dim Y]$ is a left adjoint of Φ .

If Φ is fully faithful one has an isomorphism of functors

$$\Psi \circ \Phi \cong \text{Id}_{D(Y)}[\dim Y - \dim \mathcal{P}].$$

The reason for the apparently strange choice of shift in the definition of \mathcal{Q} is to give it a good chance of being a sheaf. For example when \mathcal{P} is a vector bundle, so is \mathcal{Q} .

2.4. Assume that \mathcal{P} is a sheaf on $X \times Y$, flat over both factors. Then Φ preserves families of sheaves. In detail, let S be a scheme and \mathcal{E} an S -flat sheaf on $Y \times S$. Then

$$U = \{s \in S : \mathcal{E}_s \text{ is } \Phi\text{-WIT}_i\}$$

is the set of points of an open subscheme of S . Furthermore there exists a U -flat sheaf \mathcal{F} on $X \times U$, such that for all $s \in U$, $\mathcal{F}_s = \Phi^i(\mathcal{E}_s)$. The proof of this result is identical to that of [14], Theorem 1.6.

2.5. Suppose now that Φ is fully faithful, and take sheaves A and B on Y , with A Φ -WIT $_a$ and B Φ -WIT $_b$. Then for all i , one has

$$\text{Hom}_{D(Y)}(A, B[i]) = \text{Hom}_{D(X)}(\hat{A}[-a], \hat{B}[i - b]).$$

Rewriting this gives the identity

$$(2.5.1) \quad \text{Ext}_Y^i(A, B) = \text{Ext}_X^{i+a-b}(\hat{A}, \hat{B}),$$

which is referred to as the Parseval theorem. As a special case, note that if A is simple then so is \hat{A} .

2.6. Assume that X and Y have the same dimension. Recall that a Y -flat sheaf \mathcal{P} on $X \times Y$ is said to be *strongly simple* over Y if \mathcal{P}_y is simple for all $y \in Y$, and if for any pair y_1, y_2 of distinct points of Y and any integer i one has $\text{Ext}_X^i(\mathcal{P}_{y_1}, \mathcal{P}_{y_2}) = 0$.

Theorem 2.1. *Let \mathcal{P} be a Y -flat sheaf on $X \times Y$. Then Φ is fully faithful iff \mathcal{P} is strongly simple over Y . If \mathcal{P} is flat over X and Y then Φ is an equivalence iff \mathcal{P} is strongly simple over both factors.* \square

The main idea behind the proof was given by Mukai ([13], Theorem 4.9) and the result has appeared in various forms since then. The most general statement is due to Bondal and Orlov ([6], Theorem 1.1).

Following Orlov we shall make essential use of the following lemma. It is an immediate consequence of [5], Lemma 3.1.

Lemma 2.2. *Suppose Φ is fully faithful. Then Φ is an equivalence iff for any object E of $D(X)$, $\Psi(E) \cong 0$ implies $E \cong 0$.* \square

3. FOURIER-MUKAI TRANSFORMS FOR ELLIPTIC CURVES

Here we illustrate the results of the last section by considering the case when X and Y are elliptic curves.

3.1. Let X be an elliptic curve. Given a sheaf E on X we write its Chern class as a pair of integers $(r(E), d(E))$. Let a and b be coprime integers with $a > 0$ and let Y be the moduli space of stable bundles on X of Chern class (a, b) . In fact it is a consequence of the work of Atiyah ([2], Theorem 7) that Y is isomorphic to X ; we preserve the distinction for clarity. Let \mathcal{P} be a tautological bundle on $X \times Y$, and put

$$\Phi = \Phi_{Y \rightarrow X}^{\mathcal{P}}, \quad \Psi = \Phi_{X \rightarrow Y}^{\mathcal{P}^\vee}.$$

As we noted in 2.3, $\Psi[1]$ is a left adjoint of Φ .

Proposition 3.1. *The functor Φ is an equivalence.*

Proof. First note that \mathcal{P} is strongly simple over Y , since for any pair P_1, P_2 of non-isomorphic stable bundles on X with the same Chern class, Serre duality gives

$$\text{Ext}_X^1(P_2, P_1) = \text{Hom}_X(P_1, P_2)^\vee = 0.$$

It follows from Theorem 2.1 that Φ is fully faithful, so $\Psi \circ \Phi \cong \text{Id}_{D(Y)}[-1]$.

Next observe that the group homomorphism $\text{ch}(\Phi)$ must be an isomorphism, since

$$\text{ch}(X) \cong \text{ch}(Y) = \mathbb{Z} \oplus \mathbb{Z},$$

and $\text{ch}(\Psi) \circ \text{ch}(\Phi) = -\text{Id}_{\text{ch}(Y)}$.

To complete the proof use Lemma 2.2. Suppose E is an object of $D(X)$ such that $\Psi(E) \cong 0$. Consider the hypercohomology spectral sequence

$$E_2^{p,q} = \Psi^p(\mathcal{H}^q(E)) \implies \Psi^{p+q}(E) = 0.$$

Since $E_2^{p,q} = 0$ unless $p = 0$ or 1 , the spectral sequence degenerates at the $r = 2$ level. It follows that $\Psi(\mathcal{H}^q(E)) = 0$ for all q . But then, since $\text{ch}(\Psi)$ is an isomorphism, one has that $\mathcal{H}^q(E) = 0$ for all q , so $E \cong 0$. \square

The group homomorphism $\text{ch}(\Phi)$ is invertible and takes $(0, 1)$ to (a, b) , so must be given by some matrix

$$\begin{pmatrix} c & a \\ d & b \end{pmatrix}$$

where c and d are integers satisfying $bc - ad = \pm 1$. Then $\text{ch}(\Psi)$, which is the inverse of $-\text{ch}(\Phi)$, takes $(0, 1)$ to $\pm(a, -c)$. Since Ψ is given by a sheaf on $X \times Y$, we must take the positive sign, so that in fact $bc - ad = 1$.

This condition does not define c and d uniquely: we may replace them by $c + na$ and $d + nb$ for any integer n . This corresponds to twisting \mathcal{P} by the pull-back of a line bundle of degree n on Y . By varying n we obtain all possible values of c and d .

Theorem 3.2. *Let X be an elliptic curve and take an element*

$$A = \begin{pmatrix} c & a \\ d & b \end{pmatrix} \in SL_2(\mathbb{Z}),$$

such that $a > 0$. Then there exist vector bundles on $X \times X$ which are strongly simple over both factors, and which restrict to give bundles of Chern class (a, c) on the first factor and (a, b) on the second. For any such bundle \mathcal{P} , the resulting functor $\Phi = \Phi_{X \rightarrow X}^{\mathcal{P}}$ is an equivalence, and satisfies

$$\begin{pmatrix} r(\Phi E) \\ d(\Phi E) \end{pmatrix} = \begin{pmatrix} c & a \\ d & b \end{pmatrix} \begin{pmatrix} r(E) \\ d(E) \end{pmatrix},$$

for all objects E of $D(X)$. □

For the usual Fourier-Mukai transform \mathcal{F} on X (see [11]) one has

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The equivalences we have found are not essentially new, since one can check that they can all be obtained from composites of the functors \mathcal{F} and $L \otimes (-)$ for line bundles L on X . Later, however, we shall try to apply the transforms on each fibre of an elliptic surface, and this will only be possible for certain choices of A .

3.2. We conclude by showing that the transforms take simple sheaves to simple sheaves.

Proposition 3.3. *Let X be an elliptic curve and \mathcal{P} a bundle on $X \times X$, strongly simple over one factor. Then $\Phi = \Phi_{X \rightarrow X}^{\mathcal{P}}$ is an equivalence. Furthermore any simple sheaf E on X is Φ -WIT and the transform \hat{E} is a simple sheaf.*

Proof. The argument of Proposition 3.1 shows that Φ is an equivalence, so defining Ψ as above, there is an isomorphism

$$\Psi \circ \Phi \cong \text{Id}_{D(X)}[-1],$$

and hence, for any sheaf E on X , a spectral sequence

$$E_2^{p,q} = \Psi^p(\Phi^q(E)) \implies \begin{cases} E & \text{if } p + q = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Now $E_2^{p,q} = 0$ unless $0 \leq p, q \leq 1$, so this gives a short exact sequence

$$0 \longrightarrow \Psi^1(\Phi^0(E)) \longrightarrow E \longrightarrow \Psi^0(\Phi^1(E)) \longrightarrow 0,$$

together with the information that $\Phi^0(E)$ is Ψ -WIT₁ and $\Phi^1(E)$ is Ψ -WIT₀. The Parseval theorem then implies that

$$\mathrm{Ext}_X^1(\Psi^0(\Phi^1(E)), \Psi^1(\Phi^0(E))) = 0,$$

so E is given by a trivial extension. If E is simple it follows that one of the two sheaves $\Psi^1(\Phi^0(E))$ or $\Psi^0(\Phi^1(E))$ is zero, and E is Φ -WIT. The transform \hat{E} is then simple, as we noted in 2.5. \square

Remark 3.4. A straightforward application of Serre duality shows that a simple sheaf on X is either a stable vector bundle or the skyscraper sheaf of a point of X . In particular, a vector bundle on X is simple iff it is stable.

4. THE ELLIPTIC SURFACES $J_X(a, b)$

In this section we introduce the elliptic surfaces $J_X(a, b)$ mentioned in the introduction. By an elliptic surface we shall mean a smooth variety X of dimension 2 together with a smooth curve C and a relatively minimal morphism $\pi : X \rightarrow C$ whose general fibre is an elliptic curve. We often abuse notation and refer to X as an elliptic surface, or an elliptic surface over C , and take the morphism π as given.

4.1. Let $X \xrightarrow{\pi} C$ be an elliptic surface. Recall ([3], V.12.3) that the canonical bundle of X takes the form

$$\omega_X = \pi^*(L) \otimes \mathcal{O}_X\left(\sum (m_i - 1)f_i\right),$$

where L is a line bundle on C and m_1f_1, \dots, m_kf_k are the multiple fibres of π . This formula depends on the assumption that π is relatively minimal.

We denote the algebraic equivalence class of a fibre of π by f , and for any object E of $D(X)$ define the *fibre degree* of E to be

$$d(E) = c_1(E) \cdot f.$$

Note that the restriction of a sheaf E on X to a general fibre of π has Chern class $(r(E), d(E))$. We say that a sheaf E on X is a *fibre sheaf* if $r(E) = d(E) = 0$, or equivalently if the support of E is contained in the union of finitely many fibres of π . In this case $E \otimes \omega_X$ has the same Chern class as E , and for any other sheaf F on X ,

$$(4.1.1) \quad \chi(E, F) = \chi(F, E).$$

Let λ_X denote the highest common factor of the fibre degrees of sheaves on X . Equivalently λ_X is the smallest positive integer such that there is a divisor σ on X with $\sigma \cdot f = \lambda_X$. Note that, by Riemann-Roch, given a divisor of positive fibre degree, we can add a large multiple of f and obtain an effective divisor of the same fibre degree.

4.2. Let $X \xrightarrow{\pi} C$ be an elliptic surface and fix integers $a > 0$ and b , with $a\lambda_X$ coprime to b . Take a polarization of X of fibre degree coprime to b . By the results of [18], there exists a relative moduli scheme $\mathcal{M}(X/C) \rightarrow C$, whose points represent stable pure-dimension 1 sheaves on fibres of π .

Definition 4.1. Let $J_X(a, b)$ be the union of those components of $\mathcal{M}(X/C)$ which contain a point representing a rank a , degree b vector bundle on a non-singular fibre of π . Let $\hat{\pi}$ denote the natural map $\hat{\pi} : J_X(a, b) \rightarrow C$.

First note that the coprimality assumptions we have made imply that $Y = J_X(a, b)$ is a fine moduli scheme. Thus Y is a projective scheme whose points all represent strictly stable sheaves. Furthermore, by an argument of Mukai ([13], Theorem A.6), there is a tautological sheaf \mathcal{P} on $X \times_C Y$, such that for each point $y \in Y$, the stable sheaf corresponding to y is given by \mathcal{P}_y , the restriction of \mathcal{P} to $X_{\hat{\pi}(y)} \times \{y\}$.

Let U be the set of points $p \in C$ such that the fibre X_p is non-singular. The fibre of $\hat{\pi}$ over a point $p \in U$ is the moduli space of rank a , degree b stable sheaves on X_p , which, as we noted in section 3, is isomorphic to X_p . Thus $\hat{\pi}$ is an elliptic fibration. Clearly $\hat{\pi}$ is dominant, hence surjective, so there is some component of Y which contains sheaves supported on every fibre of π . Any other component of Y must contain a sheaf supported on a non-singular fibre, but the fibre of $\hat{\pi}$ over every point of U is connected. It follows that Y is connected.

Now let $\mathcal{M}(X)$ denote the moduli space of stable pure-dimension 1 sheaves on X , and let Z be the union of those components of $\mathcal{M}(X)$ which contain a rank a , degree b sheaf supported on a non-singular fibre of π . Thus points of Z correspond to strictly stable sheaves of Chern class $(0, af, -b)$.

There is a natural ‘extension by zero’ morphism $i : Y \rightarrow Z$, which maps a point $y \in Y$ representing the stable sheaf \mathcal{P}_y on the fibre $X_{\hat{\pi}(y)}$, to the point $z \in Z$ representing the stable sheaf on X obtained by extending \mathcal{P}_y by zero. This morphism i induces a bijection on points, since every stable sheaf of Chern class $(0, af, -b)$ is supported on some fibre of π (the support of a stable sheaf must be connected). I claim that Z is a non-singular projective surface; it will follow from this that i is an isomorphism and that Y is an elliptic surface over C .

Given a point $y \in Y$ we shall identify the sheaf \mathcal{P}_y with its extension by zero on X . If $y \in Y$ is such that \mathcal{P}_y is supported on a non-singular fibre of π , then

$$(4.2.1) \quad \mathcal{P}_y = \mathcal{P}_y \otimes \omega_X,$$

because the restriction of ω_X to any non-singular fibre of π is trivial. By EGA III.7.7.8 the dimension of the space

$$\mathrm{Hom}_X(\mathcal{P}_y, \mathcal{P}_y \otimes \omega_X)$$

is upper semi-continuous on Y , so for all $y \in Y$ there is a non-zero morphism $\mathcal{P}_y \rightarrow \mathcal{P}_y \otimes \omega_X$. But both these sheaves are stable with the same Chern class, so they are isomorphic and 4.2.1 holds for all $y \in Y$.

The Riemann-Roch formula gives

$$\chi(\mathcal{P}_y, \mathcal{P}_y) = -(af)^2 = 0,$$

so the Zariski tangent space to Z at a point $i(y)$, which is given by

$$\mathrm{Ext}_X^1(\mathcal{P}_y, \mathcal{P}_y)$$

(see e.g. [19]), always has dimension 2. Now Y fibres over C with elliptic fibres, so has dimension at least 2, and it then follows that Z is a non-singular projective surface as claimed.

Extending our tautological sheaf \mathcal{P} by zero, we obtain a sheaf on $X \times Y$ which we shall also denote by \mathcal{P} , such that for each point $y \in Y$, \mathcal{P}_y is a stable sheaf of Chern class $(0, af, -b)$ on X .

For any two distinct points y_1, y_2 of Y , Serre duality implies that

$$\mathrm{Ext}_X^2(\mathcal{P}_{y_1}, \mathcal{P}_{y_2}) = \mathrm{Hom}_X(\mathcal{P}_{y_2}, \mathcal{P}_{y_1})^\vee = 0,$$

and since $\chi(\mathcal{P}_{y_1}, \mathcal{P}_{y_2}) = 0$, this is enough to show that \mathcal{P} is strongly simple over Y . By Theorem 2.1, the functor $\Phi = \Phi_{Y \rightarrow X}^{\mathcal{P}}$ is fully faithful.

Proposition 4.2. *The scheme $Y = J_X(a, b)$ is an elliptic surface over C . Furthermore, the sheaf \mathcal{P} is strongly simple over Y , so $\Phi = \Phi_{Y \rightarrow X}^{\mathcal{P}}$ is fully faithful.*

Proof. It only remains to show that Y is relatively minimal over C . Suppose not, i.e. that there exists a (-1) -curve D contained in a fibre of $\hat{\pi}$. Then $\mathcal{K}_Y \cdot D < 0$, so

$$\chi(\mathcal{O}_D, \mathcal{O}_Y) = \chi(\omega_Y|_D) \neq \chi(\mathcal{O}_D) = \chi(\mathcal{O}_Y, \mathcal{O}_D).$$

Since Φ is fully faithful this implies that $\chi(E, F) \neq \chi(F, E)$, where $E = \Phi(\mathcal{O}_D)$ and $F = \Phi(\mathcal{O}_Y)$. But for each i , $\mathcal{H}^i(E)$ is a fibre sheaf (because \mathcal{O}_D is), so this contradicts 4.1.1. \square

Remark 4.3. Suppose we use two different polarizations of X to define elliptic surfaces $J_X(a, b)$ and $J'_X(a, b)$ over C . Then, since the stability of a sheaf on a smooth curve does not depend on a choice of polarization, the two spaces will be isomorphic over the open subset U considered above, and hence birational. Since both are relatively minimal over C , [3], Proposition III.8.4 implies that they are isomorphic as elliptic surfaces over C .

4.3. In the next section we show that the functor Φ is an equivalence. For now, let us note that the restriction of Φ to a non-singular fibre of π yields one of the transforms considered in section 3. Indeed, if $p \in C$ and $i_p : X_p \hookrightarrow X$ and $j_p : Y_p \hookrightarrow Y$ are the inclusion of the non-singular fibres X_p and Y_p , then a simple base-change (see [6], Lemma 1.3) gives an isomorphism of functors

$$\mathbf{L}i_p^* \circ \Phi \cong \Phi_p \circ \mathbf{L}j_p^*.$$

Here Φ_p is the functor $\Phi_{Y_p \rightarrow X_p}^{\mathcal{P}_p}$ and \mathcal{P}_p , the restriction of \mathcal{P} to $X_p \times Y_p$, is a tautological bundle parameterising stable bundles on X_p of rank a and degree b . Thus Φ_p coincides with one of the transforms of Theorem 3.2, and in particular there is a matrix

$$\begin{pmatrix} c & a \\ d & b \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}),$$

such that for all objects E of $D(Y)$,

$$(4.3.1) \quad \begin{pmatrix} r(\Phi E) \\ d(\Phi E) \end{pmatrix} = \begin{pmatrix} c & a \\ d & b \end{pmatrix} \begin{pmatrix} r(E) \\ d(E) \end{pmatrix}.$$

Furthermore, Proposition 3.3 gives

Lemma 4.4. *Let E be a Φ -WIT sheaf on Y whose restriction to the general fibre of $\hat{\pi}$ is simple. Then the restriction of \hat{E} to the general fibre of π is also simple. \square*

5. FOURIER-MUKAI TRANSFORMS FOR ELLIPTIC SURFACES

Here we prove Theorem 1.2. Let $X \xrightarrow{\pi} C$ be an elliptic surface, fix integers $a > 0$ and b , with $a\lambda_X$ coprime to b , and let Y denote the elliptic surface $\hat{\pi} : J_X(a, b) \rightarrow C$ defined in the last section. Fix a tautological sheaf on $X \times_C Y$, and extend by zero to obtain a sheaf \mathcal{P} on $X \times Y$. Let \mathcal{Q} be the object $(\mathcal{P}^\vee \otimes \pi_X^* \omega_X)[1]$ of $D(X \times Y)$, and define functors

$$\Phi = \Phi_{Y \rightarrow X}^{\mathcal{P}}, \quad \Psi = \Phi_{X \rightarrow Y}^{\mathcal{Q}}.$$

As we noted in 2.3, $\Psi[1]$ is a left adjoint of Φ .

Lemma 5.1. *The object \mathcal{Q} is a sheaf on $X \times Y$. Moreover, \mathcal{P} and \mathcal{Q} are both flat over X and Y .*

Proof. For each point $(x, y) \in X \times Y$, consider the commutative diagram

$$\begin{array}{ccc} \text{Spec } \mathbb{C} & \xrightarrow{j_x} & X \\ j_y \downarrow & & i_y \downarrow \\ Y & \xrightarrow{i_x} & X \times Y \end{array}$$

where j_x and i_x are the inclusions of $\{x\}$ in X and $\{x\} \times Y$ in $X \times Y$ respectively. Similarly for j_y and i_y .

First note that by an argument of Mukai ([12], p. 105), any pure-dimension 1 sheaf on any surface has a locally-free resolution of length 2. This implies that for all $y \in Y$, $(\mathcal{P}_y)^\vee[1]$ is a sheaf on X . But

$$\mathbf{L}i_y^*(\mathcal{P}^\vee[1]) = (\mathbf{L}i_y^*(\mathcal{P}))^\vee[1] = (\mathcal{P}_y)^\vee[1],$$

so the corresponding hypercohomology spectral sequence implies that $\mathcal{P}^\vee[1]$ is a sheaf on $X \times Y$, flat over Y (see [6], Proposition 1.5).

To show that \mathcal{P} is flat over X consider the spectral sequence

$$E_{p,q}^2 = \mathbf{L}_p j_y^*(\mathbf{L}_q i_x^*(\mathcal{P})) \implies \mathbf{L}_{(p+q)} j_x^*(\mathcal{P}_y).$$

Here $\mathbf{L}_p f^*(E)$ denotes the $(-p)$ th cohomology sheaf of $\mathbf{L}f^*(E)$. Since \mathcal{P}_y has a two-step resolution, the right-hand side is non-zero only if $p + q = 0$ or 1, so one concludes that

$$\mathbf{L}_1 j_y^*(\mathbf{L}_1 i_x^*(\mathcal{P})) = 0,$$

for all $y \in Y$. This implies that $\mathbf{L}_1 i_x^*(\mathcal{P})$ is locally free on Y . But for any $x \in X$ one can find $y \in Y$ such that (x, y) does not lie in the support of \mathcal{P} , so $\mathbf{L}_1 i_x^*(\mathcal{P}) = 0$ for all $x \in X$, and \mathcal{P} is flat over X .

Finally, the isomorphism

$$\mathbf{L}i_x^*(\mathcal{P}^\vee[1]) \cong (\mathcal{P}_x)^\vee[1],$$

implies that both sides are sheaves on Y , so $\mathcal{P}^\vee[1]$ is flat over X . \square

The next lemma shows that the relationship between X and Y is entirely symmetrical.

Lemma 5.2. *There exists an integer c such that $X \cong J_Y(a, c)$.*

Proof. If X_p is a non-singular fibre of π then the restriction of \mathcal{P} to $X_p \times Y_p$ is a tautological bundle parameterising stable bundles on X_p . By the results of section 3, this bundle is strongly simple over both factors, so for any point $x \in X$ lying on a non-singular fibre of π , the sheaf \mathcal{P}_x is a stable sheaf on Y . Let its Chern class be $(0, af, -c)$. I claim that c is coprime to $a\lambda_Y$, so that $J_Y(a, c)$ is well-defined. Assuming this for the moment, note that as in Remark 4.3, the two elliptic surfaces X and $J_Y(a, c)$ over C are isomorphic away from the singular fibres, so are isomorphic.

Since the object \mathcal{Q}_x of $D(Y)$ has Chern class $(0, af, c)$, to prove the claim it will be enough to exhibit an object E of $D(Y)$ such that $\chi(\mathcal{Q}_x, E) = 1$. But this is possible by 4.3.1, since the result of section 2.3 implies that

$$\chi(\mathcal{Q}_x, E) = -\chi(\mathbb{C}_x, \Phi E) = -r(\Phi E),$$

for any object E of $D(Y)$. \square

We can now prove Theorem 1.2. By Proposition 4.2, Φ is fully faithful, so $\Psi \circ \Phi \cong \text{Id}_{D(Y)}[-1]$. It follows that $\text{ch}(\Psi) \circ \text{ch}(\Phi) = -\text{Id}_{\text{ch}(Y)}$, and $\text{ch}(\Phi)$ embeds $\text{ch}(Y)$ as a direct summand of $\text{ch}(X)$. Applying Lemma 5.2, we can repeat the argument and obtain $\text{ch}(X)$ as a direct summand of $\text{ch}(Y)$. This shows that $\text{ch}(\Phi)$ is an isomorphism.

We complete the proof by applying Lemma 2.2. Suppose E is an object of $D(X)$ such that $\Psi(E) \cong 0$. Consider the hypercohomology spectral sequence

$$E_2^{p,q} = \Psi^p(\mathcal{H}^q(E)) \implies \Psi^{p+q}(E) = 0.$$

Since \mathcal{Q} is supported on $X \times_C Y$ and is flat over X and Y , one has that $E_2^{p,q} = 0$ unless $p = 0$ or 1 . It follows that the spectral sequence degenerates at the $r = 2$ level, so $\Psi(\mathcal{H}^q(E)) = 0$ for all q . But since $\text{ch}(\Psi)$ is an isomorphism, and a non-zero sheaf has non-zero Chern character, this implies that $\mathcal{H}^q(E) = 0$ for all q , so $E \cong 0$.

We summarise our results in the following theorem.

Theorem 5.3. *Let $X \xrightarrow{\pi} C$ be an elliptic surface and take an element*

$$\begin{pmatrix} c & a \\ d & b \end{pmatrix} \in SL_2(\mathbb{Z}),$$

such that λ_X divides d and $a > 0$. Let Y be the elliptic surface $J_X(a, b)$ over C . Then there exist sheaves \mathcal{P} on $X \times Y$, flat and strongly simple over both factors such that for any point $(x, y) \in X \times Y$, \mathcal{P}_y has Chern class $(0, af, -b)$ on X and \mathcal{P}_x has Chern class $(0, af, -c)$ on Y .

For any such sheaf \mathcal{P} , the resulting functor $\Phi = \Phi_{Y \rightarrow X}^{\mathcal{P}}$ is an equivalence and satisfies

$$(5.0.2) \quad \begin{pmatrix} r(\Phi E) \\ d(\Phi E) \end{pmatrix} = \begin{pmatrix} c & a \\ d & b \end{pmatrix} \begin{pmatrix} r(E) \\ d(E) \end{pmatrix},$$

for all objects E of $D(Y)$.

Proof. Take a tautological sheaf \mathcal{P} on $X \times Y$ and put $\Phi = \Phi_{Y \rightarrow X}^{\mathcal{P}}$. As we showed above, Φ is an equivalence and there exist integers c and d such that 5.0.2 holds. Now λ_X divides $d(\Phi E)$ for any object E of $D(Y)$, so λ_X divides λ_Y and d . By symmetry $\lambda_X = \lambda_Y$. As in section 3, c and d are not uniquely defined: we can replace them by $c + n\lambda_X a$ and $d + n\lambda_X b$ by twisting \mathcal{P} by the pull-back of a line bundle of fibre degree $n\lambda_X$ on Y . \square

Remark 5.4. As a corollary of the proof of Theorem 5.3, note that we can always choose \mathcal{P} so that for some polarization of X , \mathcal{P}_y is stable for all $y \in Y$. By Lemma 5.2, we could also view X as a moduli space of sheaves on Y , and take \mathcal{P} such that for some polarization of Y , $\mathcal{P}_x^\vee[1]$ is stable for all $x \in X$.

6. PROPERTIES OF THE TRANSFORMS

Let $X \xrightarrow{\pi} C$ be an elliptic surface, fix an element

$$\begin{pmatrix} c & a \\ d & b \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}),$$

with $a > 0$ and λ_X dividing d , let Y be the elliptic surface $J_X(a, b)$ and take a sheaf \mathcal{P} on $X \times Y$ as in Theorem 5.3. As in the last section, define the sheaf $\mathcal{Q} = (\mathcal{P}^\vee \otimes \pi_X^* \omega_X)[1]$, and the functors Φ and Ψ . Since Φ is an equivalence, one has isomorphisms

$$(6.0.3) \quad \Psi \circ \Phi \cong \mathrm{Id}_{D(Y)}[-1], \quad \Phi \circ \Psi \cong \mathrm{Id}_{D(X)}[-1].$$

In this section we give some properties of the transforms which will be useful in section 7. Note that, because of the symmetry of the situation, for each result we give here, there will be another result obtained by exchanging Φ and Ψ , and X and Y .

6.1. The functor Φ is left exact, because \mathcal{P} is flat over Y . Thus given a short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0,$$

one obtains a long exact sequence

$$(6.1.1) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \Phi^0(A) & \longrightarrow & \Phi^0(B) & \longrightarrow & \Phi^0(C) \\ & & \longrightarrow & & \Phi^1(A) & \longrightarrow & \Phi^1(B) & \longrightarrow & \Phi^1(C) & \longrightarrow & 0. \end{array}$$

6.2. The isomorphisms 6.0.3 imply that if E is a Ψ -WIT $_i$ sheaf on X ($i = 0, 1$), then \hat{E} is a Φ -WIT $_{1-i}$ sheaf on Y . More generally, for any sheaf E on X there is a spectral sequence

$$E_2^{p,q} = \Phi^p(\Psi^q(E)) \implies \begin{cases} E & \text{if } p+q=1 \\ 0 & \text{otherwise.} \end{cases}$$

Since $E_2^{p,q} = 0$ unless $0 \leq p, q \leq 1$, this yields a short exact sequence

$$0 \longrightarrow \Phi^1(\Psi^0(E)) \longrightarrow E \longrightarrow \Phi^0(\Psi^1(E)) \longrightarrow 0,$$

together with the information that $\Psi^0(E)$ is Φ -WIT $_1$ and $\Psi^1(E)$ is Φ -WIT $_0$. Note that 6.0.3 then implies that $\Phi^1(\Psi^0(E))$ is Ψ -WIT $_0$ and $\Phi^0(\Psi^1(E))$ is Ψ -WIT $_1$.

Lemma 6.1. *For any sheaf E on X there is a unique short exact sequence*

$$0 \longrightarrow A \longrightarrow E \longrightarrow B \longrightarrow 0,$$

such that A is Ψ -WIT₀ and B is Ψ -WIT₁.

Proof. For the uniqueness, suppose there is another such sequence

$$0 \longrightarrow A' \longrightarrow E \longrightarrow B' \longrightarrow 0.$$

Then, since A' is Ψ -WIT₀ and B is Ψ -WIT₁, the Parseval theorem implies that there is no non-zero map $A' \rightarrow B$, so the inclusion of A' in E factors through A . By symmetry $A = A'$. \square

6.3. Given a torsion-free sheaf E on X , put $\mu(E) = d(E)/r(E)$.

Lemma 6.2. *Let E be a torsion-free sheaf on X . If E is Ψ -WIT₀ then $\mu(E) \geq b/a$. Similarly if E is Ψ -WIT₁ then $\mu(E) \leq b/a$.*

Proof. If E is Ψ -WIT₁ then $\Psi(E)[1]$ is a sheaf so $r(\Psi E) \leq 0$. Similarly, if E is Ψ -WIT₀ then $r(\Psi E) \geq 0$. Since $\Psi[1]$ is the inverse of Φ , one has

$$\begin{pmatrix} r(\Psi E) \\ d(\Psi E) \end{pmatrix} = \begin{pmatrix} -b & a \\ d & -c \end{pmatrix} \begin{pmatrix} r(E) \\ d(E) \end{pmatrix},$$

for any object E of $D(X)$. The result follows. \square

A similar argument gives

Lemma 6.3. *Let T be a Ψ -WIT₁ torsion sheaf on X . Then T is a fibre sheaf.* \square

Combining Lemma 6.2 with Lemma 6.1 we obtain

Lemma 6.4. *Let E be a torsion-free sheaf on X such that the restriction of E to the general fibre of π is stable. Suppose $\mu(E) < b/a$. Then E is Ψ -WIT₁.*

Proof. Consider the short exact sequence of Lemma 6.1. If A is non-zero, it is torsion-free and one has $\mu(A) \geq b/a > \mu(E)$. Restricting to the general fibre of π this gives a contradiction. Hence $A = 0$ and E is Ψ -WIT₁. \square

6.4. The final result we shall need is

Lemma 6.5. *A sheaf F on Y is Φ -WIT₀ iff*

$$\mathrm{Hom}_Y(F, \mathcal{Q}_x) = 0 \quad \forall x \in X.$$

Proof. First note that $\mathcal{Q}_x = \Psi(\mathbb{C}_x)$ is Φ -WIT₁. If F is Φ -WIT₀, then the Parseval theorem implies that there are no non-zero maps $F \rightarrow \mathcal{Q}_x$.

Conversely, if F is not Φ -WIT₀, then by the argument of Lemma 6.1, there is a surjection $F \rightarrow B$ with B a Φ -WIT₁ sheaf. Applying the Parseval theorem again gives

$$\mathrm{Hom}_Y(B, \mathcal{Q}_x) = \mathrm{Hom}_X(\hat{B}, \mathbb{C}_x).$$

Since \hat{B} is non-zero, there exists an $x \in X$ and a non-zero map $B \rightarrow \mathcal{Q}_x$, hence a non-zero map $F \rightarrow \mathcal{Q}_x$. \square

7. APPLICATION TO MODULI OF STABLE SHEAVES

In this section we use the relative transforms we have developed to prove Theorem 1.1.

7.1. Let $X \xrightarrow{\pi} C$ be an elliptic surface and fix a triple

$$(r, \Lambda, k) \in \mathbb{N} \times \text{NS}(X) \times \mathbb{Z},$$

such that r is coprime to $d = \Lambda \cdot f$. The proof of the following result is entirely analogous to the rank 2 case ([7], Theorem I.3.3) so we omit it (see also [15], Proposition I.1.6).

Proposition 7.1. *There exist polarizations of X with respect to which a torsion-free sheaf E on X with Chern class (r, Λ, k) is μ -stable whenever it is μ -semi-stable, and this is the case iff the restriction of E to all but finitely many fibres of π is stable. \square*

Taking such a polarization, define $\mathcal{M} = \mathcal{M}_X(r, \Lambda, k)$ to be the (fine) moduli space of stable torsion-free sheaves on X of Chern class (r, Λ, k) . We identify the closed points of \mathcal{M} with the stable sheaves which they represent. As in the rank 2 case ([7], Lemma III.3.6), one shows that for any $E \in \mathcal{M}$,

$$\text{Ext}_X^2(E, E) = H^2(X, \mathcal{O}_X).$$

It then follows from the general results of [1] that \mathcal{M} , if non-empty, is smooth of dimension $\dim(\text{Pic}^\circ(X)) + 2t$, where

$$2t = 2rk - (r-1)\Lambda^2 - (r^2-1)\chi(\mathcal{O}_X),$$

and that if $t < 0$ then \mathcal{M} is empty. In what follows we take $r > 1$ and assume that t is non-negative.

Let a and b be the unique pair of integers satisfying $br - ad = 1$, with $0 < a < r$. Let $\hat{\pi} : Y \rightarrow C$ be the elliptic surface $J_X(a, b)$ and put

$$\mathcal{N} = \mathcal{M}_Y(1, 0, t) = \text{Pic}^\circ(Y) \times \text{Hilb}^t(Y).$$

We shall prove Theorem 1.1 by showing that \mathcal{M} is birationally equivalent to \mathcal{N} .

7.2. Let \mathcal{P} be a sheaf on $X \times Y$ as in Theorem 5.3, with matrix

$$\begin{pmatrix} r & a \\ d & b \end{pmatrix},$$

and define equivalences of categories Φ and Ψ as in section 6. As we noted in Remark 5.4, we can assume that we have chosen \mathcal{P} , and a polarization of Y , so that \mathcal{Q}_x is a stable sheaf for all $x \in X$.

Take a sheaf E on X of Chern class (r, Λ, k) . The formula given in the proof of Lemma 6.2 shows that $\Psi(E)$ has rank 1 and fibre degree 0. Twisting \mathcal{P} by the pull-back of a line bundle on Y we can assume that $c_1(\Psi E) = 0$, and the formula 2.5.1, together with Riemann-Roch then implies that $\Psi(E)$ has Chern class $(1, 0, t)$.

Note that, by Lemma 6.4, any element E of \mathcal{M} is Ψ -WIT₁. Define

$$\mathcal{U} = \{E \in \mathcal{M} : \hat{E} \text{ is torsion-free}\}.$$

By the result of section 2.4, \mathcal{U} is an open subscheme of \mathcal{M} . Also define the open subscheme

$$\mathcal{V} = \{F \in \mathcal{N} : F \text{ is } \Phi\text{-WIT}_0\}.$$

Lemma 7.2. *The transform Φ gives an isomorphism between the schemes \mathcal{U} and \mathcal{V} .*

Proof. For any point $E \in \mathcal{U}$, E is Ψ -WIT₁ and $\hat{E} \in \mathcal{V}$. Suppose now that $F \in \mathcal{V}$ and put $E = \hat{F}$. Claim that $E \in \mathcal{U}$. By Lemma 4.4 the restriction of E to the general fibre of X is stable, so it is only necessary to check that E is torsion-free. Suppose E has a torsion subsheaf T . Then since E is Ψ -WIT₁, the long exact sequence 6.1.1 implies that T is Ψ -WIT₁ also, hence, by Lemma 6.3, a fibre sheaf. Applying Ψ gives a sequence

$$0 \longrightarrow \Psi^0(E/T) \xrightarrow{f} \hat{T} \longrightarrow F \longrightarrow \Psi^1(E/T) \longrightarrow 0.$$

Since F is torsion-free and \hat{T} is a fibre sheaf, f must be an isomorphism. But, by the result of 6.2, $\Psi^0(E/T)$ is Φ -WIT₁, and \hat{T} is Φ -WIT₀. It follows that both sheaves are zero, so $T = 0$ and E is torsion-free.

The proof of the lemma is completed by appealing to the general result quoted in section 2.4. \square

Clearly, we need to show that \mathcal{U} and \mathcal{V} are non-empty. Take $F \in \mathcal{N}$. Then $F = L \otimes \mathcal{I}_Z$, with $L \in \text{Pic}^\circ(Y)$ and Z a zero-dimensional subscheme of Y of length t . By Lemma 6.5, F is Φ -WIT₀ precisely when there is no non-zero map $F \rightarrow \mathcal{Q}_x$ for any $x \in X$. Since \mathcal{Q}_x is supported on the fibre $Y_{\pi(x)}$ of $\hat{\pi}$, any map $F \rightarrow \mathcal{Q}_x$ factors via $F|_{Y_{\pi(x)}}$, and hence via a stable, pure dimension 1 sheaf on Y of Chern class $(0, f, s)$, where s is the number of points of Z lying on the fibre $Y_{\pi(x)}$. Now \mathcal{Q}_x has Chern class $(0, af, r)$, and is stable, so if $s < r/a$, any map $F \rightarrow \mathcal{Q}_x$ is zero. This argument, and the fact that $r > a$, gives the following results.

Lemma 7.3. *Let $F = L \otimes \mathcal{I}_Z$, with $L \in \text{Pic}^\circ(Y)$ and Z a set of t points $\{y_1, \dots, y_n\}$ lying on distinct fibres of $\pi : Y \rightarrow C$. Then F is an element of \mathcal{V} .* \square

Lemma 7.4. *If $r > at$ then $\mathcal{V} = \mathcal{N}$.* \square

Remark 7.5. Applying Φ to the short exact sequence

$$0 \longrightarrow F \longrightarrow L \longrightarrow \mathcal{O}_Z \longrightarrow 0,$$

gives a sequence

$$0 \longrightarrow \hat{F} \longrightarrow \hat{L} \longrightarrow \oplus \mathcal{P}_{y_i} \longrightarrow 0.$$

Let us assume for simplicity that X is simply-connected. Then we see that an open subset of \mathcal{M} is obtained from the fixed bundle $\hat{\mathcal{O}}_Y$ by taking t distinct non-singular fibres $\{f_1, \dots, f_t\}$ of π and stable bundles P_i of rank a and degree b on f_i , and taking the kernel of the unique morphism

$$\hat{\mathcal{O}}_Y \longrightarrow \oplus P_i.$$

Furthermore, when X is nodal, the proof of [7], Proposition III.3.11 shows that $\hat{\mathcal{O}}_Y$ is the unique sheaf (up to twists) on X whose restriction to every reduction of a fibre of π is stable.

In the rank 2 case, this corresponds to Friedman's method of constructing bundles using elementary modifications ([7], section III.3).

7.3. To complete the proof of Theorem 1.1 we must show that \mathcal{M} is irreducible, i.e. that \mathcal{M} has only one connected component. Let us suppose, for contradiction, that there is a connected component \mathcal{W} of \mathcal{M} which does not meet \mathcal{U} .

Let E be a point of \mathcal{W} . Then E is Ψ -WIT₁, and the transform \hat{E} is a sheaf of Chern class $(1, 0, t)$ on Y , with a non-zero torsion sheaf. By the argument of Lemma 4.4, the restriction of \hat{E} to the general fibre of $\hat{\pi}$ is simple, hence stable.

Lemma 7.6. *Let $n \geq 1$ be an integer. Then for a general zero-dimensional subscheme $Z \in \text{Hilb}^{rn}(Y)$, there is a unique morphism $\hat{E} \rightarrow \mathcal{O}_Z$. Furthermore, for general Z , this morphism surjects and the kernel K is Φ -WIT₀. The transform \tilde{K} is then an element of the moduli space*

$$\tilde{\mathcal{M}} = \mathcal{M}_X(r, \Lambda - (rna)f, k + rnb - rna(\Lambda \cdot f)).$$

Proof. We may suppose that Z consists of rn points lying on distinct non-singular fibres f_1, \dots, f_{rn} of $\hat{\pi}$. We can also suppose that \hat{E} is locally-free at each of the points of Z . Then there is a unique morphism $\hat{E} \rightarrow \mathcal{O}_Z$ and this map surjects, giving an exact sequence

$$0 \longrightarrow K \longrightarrow \hat{E} \longrightarrow \mathcal{O}_Z \longrightarrow 0.$$

By Lemma 6.5, to prove that K is Φ -WIT₀, we must show that there are no non-zero morphisms $K \rightarrow \mathcal{Q}_x$ for any $x \in X$. We only need to check this when \mathcal{Q}_x is supported on one of the fibres f_1, \dots, f_{rn} since the restrictions of \hat{E} and K to any other fibre are identical, and \hat{E} is Φ -WIT₀. But we can always take Z so that the restriction of \hat{E} to each of the fibres f_i is a degree 0 line bundle. This will be enough since \mathcal{Q}_x is stable of degree $-r$. \square

Twisting by $\mathcal{O}_X(anf)$ gives an isomorphism between the spaces $\tilde{\mathcal{M}}$ and $\mathcal{M}_X(r, \Lambda, k + n)$, so a general theorem of Gieseker-Li and O'Grady (see e.g. [16]) implies that for large enough n , $\tilde{\mathcal{M}}$ is irreducible. It follows from the results of 7.2 that the general element of $\tilde{\mathcal{M}}$ has torsion-free transform. Now the construction of the lemma gives a rational map

$$\theta : \mathcal{W} \times \text{Hilb}^{rn}(Y) \dashrightarrow \tilde{\mathcal{M}},$$

and since all points in the image of θ have non-torsion-free transforms, θ cannot be dominant. But we shall show below that the general fibre of θ is zero-dimensional. Since θ is a map between two varieties of the same dimension, this will give a contradiction.

Take an element of $\tilde{\mathcal{M}}$, and let K be its transform. We must show that there are only finitely many pairs

$$(E, Z) \in \mathcal{W} \times \text{Hilb}^{rn}(Y),$$

such that Z consists of rn distinct points at which \hat{E} is locally free, and $K = \hat{E} \otimes \mathcal{I}_Z$.

Given such a pair, note that Z does not meet the support of the torsion subsheaf of \hat{E} , so the torsion subsheaves T of \hat{E} and $\hat{E} \otimes \mathcal{I}_Z = K$ are equal.

Thus Z is a subset of the finite set of points at which K/T is not locally-free. This implies that the number of possible choices of Z is finite.

Finally, if we have two pairs (E_1, Z) , and (E_2, Z) then $E_1 = E_2$, because there is only one extension of K by \mathcal{O}_Z which is locally-free at each of the points of Z . This completes the proof.

7.4. We conclude with the following simple corollary of Theorem 1.1.

Corollary 7.7. *Given integers $a > 0$ and b with $a\lambda_X$ coprime to b , and a positive integer t , there are polarizations of X such that a component of the moduli space of torsion-free stable sheaves on X is isomorphic to*

$$\mathrm{Pic}^\circ(J_X(a, b)) \times \mathrm{Hilb}^t(J_X(a, b)).$$

Proof. Take $r > at$ such that br is congruent to 1 modulo $a\lambda_X$. Then there exists a divisor Λ on X such that $d = \Lambda \cdot f$ is coprime to r and $br - ad = 1$. Adding multiples of f to Λ if necessary, it is easy to check that one can choose k such that 1.1.1 holds. Applying Theorem 1.1 then gives the result. \square

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